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Market Dynamics with Non-Homogeneous Poisson Processes

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ABSTRACT

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The Bertrand Duopoly model for demand in economics is a well-used model. Although this model has important insights towards pricing strategy, it does not accurately depict true market behaviors. In this paper, we will examine the advantages and disadvantages of the current model and its assumptions.

We then take a whole new approach towards modeling this phenomena, using Poisson processes to model the demand of goods. We will discuss why this is a better approach and explain how we can extend this to better understand pricing strategies and market dynamics. We then apply our findings to the newsvendor problem, a commonly used problem in inventory management. Using non-homogeneous Poisson processes we explain how to find an optimal pricing strategy and an optimal inventory level for the newsvendor problem.

In this paper we explain how to extend the newsvendor problem to a newsvendor duopoly problem. Again we show how to find the optimal pricing strategies and inventory levels for multiple goods in a market. Having found the optimal pricing strategy and inventory level, we then examine the market dynamics in more details. We explore monopolistic and duopolistic markets where the goods range from complements to substitutes and homogeneous to differentiated goods. We discuss how to model the progression of the inventory probabilities and then explain how to price inventory options.

Keywords: Poisson rates, inventory options, inventory management, dynamic pricing, optimal inventory, newsvendor problem

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Contents

Chapter 1. Introduction

Currently the Bertrand model is well used and is a widely accepted representation of demand in economic models. Although many insights are gained by using it, in this paper, we question the assumptions and results of the Bertrand model. We propose a new model to consider demand using Poisson rates. In Section 2, we provide background information about the Bertrand model as well as the Poisson distribution. Included is the basic assumptions of the Bertrand model and basic properties of the model and also of Poisson processes.

In Section 3 we propose our new model and show how it is applied to a basic well known problem called the newsvendor problem. We give the background and intuition behind solving the posed problem and visit the work of Angelos, et. al. [1]. We show how a firm can use Poisson rates to develop an optimal pricing strategy for a good [10], find the optimal initial inventory for the retailer, price an inventory option, and find the optimal wholesaler price all in a finite time horizon sales period. We go through an example of this one good market setup.

We then generalize the newsvendor problem to include multiple retailers. We call this problem the newsvendor duopoly problem and explain how to extend the previous results to several different market types. First in Section 4, we consider a duopolistic case where the two retailers are selling the same exact product and have the same demand constraints. Amongst other things, we observe that the pricing strategy of firms selling complementary goods isn't always intuitive. We see that a firm with a lot of inventory compared to its rival is likely to drastically drop prices, even below the optimal sales price at expiration. It is also shown how cross price elasticity changes over time in a linear demand case. We also observe that when goods are substitutes, they have higher expected profits when their competitor has less inventory; when the goods are complements, the firms have higher expected profits when their competitor has more inventory. We continue to extend the framework to other

market structures.

In Section 5 we look at a duopolistic market in the presence of collusion. We explain how this is synonymous to examining a monopolistic market. Against conventional wisdom, we see that optimal initial inventory levels of monopolies aren't necessarily lower than those in the duopolistic market. We also observe that the monopolistic profits are not significantly better than the combined total in the duopolistic market nor are prices necessarily always higher in the monopolistic case.

Section 6 addresses the same problem with differentiated goods. It covers both the duopolistic and monopolistic market structures. Our conclusions about monopolies are confirmed as we see cases when the total profit is less than that of the duopolistic case and when the optimal initial inventory levels are higher in the monopolistic market. Because the goods are differentiated, there is a disparity in the leverage the two firms have in the market. The differences in market leverage causes there to be imbalances in the market. We observe several effects of the market imbalances in the behavior of the pricing strategies, the cross price elasticities, and the expected remaining revenue.

We offer our final remarks in the conclusion found in Section 7 and conclude by reviewing our findings and defending our approach towards understanding market dynamics using non-homogeneous Poisson processes.

Chapter 2. Background

2.1 Bertrand Model

The Bertrand model is an economical model used to understand the behavior of firms in an oligopolistic market. It was introduced as a rival model to the already established Cournot

model, proposed in 1838 by the French philosopher and mathematician Antoine Augustin Cournot [18]. Unlike the Cournot model where firms optimize profit by choosing output levels, firms in the Bertrand model optimize their profits by choosing their good's price.

Joseph Bertrand suggested the new model in 1883 in a critical review of Cournot's previous work. One of the critiques of Cournot's work was that changing quantity may not be feasible or practical in the short run. This led to one of the advantages of Bertrand's model, feasibility; a firm can easily and inexpensively change the prices of their goods, unlike changing output, which could take extended periods of time to make the adequate changes at the factories both at the input and output side.

One of the negative results that dissuade economists from fully accepting the Bertrand model is that price must always equal marginal cost in a multi-firm market, with few exceptions. We acknowledge the research done to show price doesn't have to equal marginal cost in the presence of product differentiation, demand fluctuations, or in an infinitely pricing repeated game but question the core assumptions in the Bertrand model. (Edgeworth showed that $p \neq MC$ in the case of diminishing returns to scale and quantity restrictions [18].)

2.1.1 Assumptions. The Bertrand model makes the following two controversial assumptions:

1. Consumers always purchase from the cheapest seller.
2. If two sellers charge the same price, purchases/sales are split evenly.

This is expressed in the demand for good i as shown below,

$$q_i = \begin{cases} 0 & \text{if } p_i > a \\ 0 & \text{if } p_i > p_j \\ \frac{a-p}{2b} & \text{if } p_i = p_j \equiv p < a \\ \frac{a-p_i}{b} & \text{if } p_i < \min\{a, p_j\} \end{cases}$$

where q_i is the quantity demanded for good i, p_i is the price of good i, and a is the absolute highest a consumer will pay for the good.

This is to suggest that the smallest monetary increase will detract all potential buyers and send them to a competitor. This is not very likely. Consider buying a car. Assume there are two dealers who sell the exact same car but one charges \$0.01 more. Although the penny increase may discourage some buyers, it is unlikely that many people would change their decision on such a small difference. This inexplicable phenomenon motivates our research and our proposed model. In our paper, we will show how our model accounts for these fault in the Bertrand model.

2.2 Poisson Rates

The Poisson distribution was introduced by the French mathematician and physicist Simeon Denis Poisson in 1837 and models the number of occurrences or events that will take place in a given time interval. One of its first and most notable applications was made by Ladislaus Bortkiewicz in his book "Das Gesetz der kleinen Zahlen," German for "The Law of Small Numbers." The late 19th century publication outlined Bortkiewicz model of Prussian soldiers deaths caused by a horse kick, or otherwise known as "Bortkiewicz's disease" [11]. Although the discovery of the Poisson distribution is sometimes given to Abraham de Moivre (although not written in such clear form as Poisson's representation), some believe it should have been called the "Bortkiewicz distribution" due to his application of it towards rare events [11].

Since its acceptance in statistical theory, the Poisson distribution has been used in many fields including demography, queuing theory, reliability engineering, and epidemiology. Poisson processes are commonly used to model a variety of things including doctors' visits, telephone calls, birth-death processes, hotel booking, radioactive decay, and website requests [7],[13].

2.2.1 Poisson Rates in Demand. Given the similarity of its other applications, the Poisson distribution is a reasonable way to model demand through modeling the arrival rates of customers. The beauty of the Poisson distribution, unlike other economic models, is that it incorporates randomness. Most economic models, including the Bertrand and Cournot model, consider demand in a deterministic approach. Encompassing the randomness of demand helps companies better understand what is happening in their respective market and allows them to make better pricing strategy decisions as well as teaching them how to adapt in order to optimize their profit.

Adopting this probabilistic approach and using Poisson rates, allows demand to fluctuate on a given time interval even in the presence of constant prices. You wouldn't expect a bakery to sell the same amount of bread everyday even if the prices remained constant, so in this regard, it does indeed make sense to use Poisson rates. It is not in the scope of this paper to explain how to approximate the Poisson parameters for a given economical system but we note that this is also an active field of research [3]. We will consider a simplistic approach by assuming the Poisson parameters for demand either follow a linear demand curve or a log-linear demand curve, both common models of demand.

2.2.2 Properties of Poisson Rates. Let X be a random variable following a Poisson distribution with Poisson parameter λ. Recall that on a small fixed interval of time Δt, the

Poisson process satisfies the following

$$\begin{aligned}\mathbb{P}(\quad X(t+\Delta t) \quad &= n+m | X(t)=n) \\ &= \begin{cases} 1-\lambda\Delta t + o(\Delta t) & \text{if } m=0 \\ \lambda\Delta t + o(\Delta t) & \text{if } m=1 \\ o(\Delta t) & \text{if } m>1 \end{cases}\end{aligned}$$

where $m, n \in \mathbb{Z}$. Therefore if we modeled demand, D, on the given interval, we could represent the number of units demanded with a certain probability as follows

$$D = \begin{cases} 0 & \text{w/ prob } 1-\lambda\Delta t + o(\Delta t) \\ 1 & \text{w/ prob } \lambda\Delta t + o(\Delta t) \\ \geq 2 & \text{w/ prob } o(\Delta t) \end{cases} \tag{2.1}$$

We will use this fact of Poisson processes repeatedly throughout the paper in order to solve subproblems of a dynamical system.

Chapter 3. Newsvendor Problem

The newsvendor problem is one of the canonical problems in inventory management [17],[15]. In the newsvendor problem, a retailer or salesman buys initial inventory, I_0 at wholesale price, C_0, at the beginning of the day, $t=0$, and sells during the day $[0,T]$. The good is perishable and so the time horizon, $[0,T]$, is finite. At the end of the sale period, $t=T$, the remaining inventory, I_T, is no longer sellable (such as a newspaper) but carries a salvage price of C_T (for example the newspaper may be recycled) [16]. Many questions arise for the salesman and are addressed in current research. What is the optimal sales price? What is the optimal initial inventory? We will answer both of these questions.

We let u be the fixed price over the sales period and X be the random variable representing

the demand for the day. We write the salesman's profit function,

$$\Pi = I_0(u - C_0) - (u - C_T)I_T$$

where I_T is the only variable because everything else is a constant but I_T depends directly on the random variable X. In fact

$$I_T = \max(I_0 - X, 0).$$

The expected profit is therefore

$$\mathbb{E}[\Pi] = I_0(u - C_0) - (u - C_T)\mathbb{E}[I_T].$$

Angelos, et. al. [1] has shown that the derivative of the remaining inventory with respect to the initial inventory is

$$\frac{d}{dI_0}\mathbb{E}[I_T] = F(I_0).$$

where F is the cumulative distribution function of the random variable X. Take the derivative of expected profit with respect to the initial inventory and solve for the optimal initial inventory.

$$\begin{aligned}
\frac{d}{dI_0}\mathbb{E}[\Pi] &= (u - C_0) - (u - C_T)\frac{d}{dI_0}\mathbb{E}[I_T] \\
&= (u - C_0) - (u - C_T)F(I_0) = 0 \\
I_0^* &= F^{-1}\left(\frac{u - C_0}{u - C_T}\right)
\end{aligned}$$

We now have the optimal initial inventory given a fixed price. In the next section we explore what would happen if we varied prices as the sales period progressed.

3.1 Pricing Strategy

Consider the same problem but now allow the newsvendor to vary the price throughout the sales period. The question of what is the optimal pricing strategy given dynamic pricing now arises. This idea of dynamic pricing for a perishable good is also an active field of research [20]. Let $\{u_j(t)\}_{j=1}^{I_0}$ be a set of functions representing the sales price at time t with j remaining units of inventory. Let $X(t)$ be a nonhomogeneous Poisson process where the customers' purchases arrive at the rate $\lambda(u_j(t))$. In order to solve for the optimal pricing strategy, we will discretize the time interval, set up a dynamical program, solve the subproblem by finding the optimal price on the given time interval, and then back iterate to get an optimal pricing strategy [4],[6],[5].

First discretize the sales period $[0, T]$ into n equal time intervals. Fix $u_j(t)$ for each time interval and let $u_{i,j} = u_j(i\Delta t)$. Also let $\lambda_{i,j} = \lambda(u_j(i\Delta t))$ and $I_i = I(i\Delta t)$. By Equation 2.1, the demand for the product on a small enough time interval, i, is given by

$$D_i = \begin{cases} 0 & \text{w/ prob } 1 - \lambda_{i,j}\Delta t + o(\Delta t) \\ 1 & \text{w/ prob } \lambda_{i,j}\Delta t + o(\Delta t) \\ \geq 2 & \text{w/ prob } o(\Delta t) \end{cases} . \tag{3.1}$$

Let S be the function representing the inventory sold during period i given by

$$S(I_i, D_i) = \min(I_i, D_i).$$

Consequently, the state equation is given by

$$I_{i+1} = I_i - S(I_i, D_i). \tag{3.2}$$

Maximizing the expected profit given a set inventory level is the same as maximizing the

expected revenue given the same inventory level. Since these are the same problems given a set amount of inventory, we can examine maximizing the total revenue as if it was the expected profit and then decide the optimal inventory numerically. The total revenue is given by

$$R = \sum_{k=0}^{n-1} u_{k,I_k} S(I_k, D_k) + C_T(I_0 - \sum_{l=0}^{n-1} D_l)^+$$

where $(\cdot)^+ = \max(0, \cdot)$ and $S(I_k, D_k)$ represents the sales at time period k. Inventory at time k is given by I_k. The remaining revenue at time i is

$$R(I_i) = \sum_{k=i}^{n-1} u_{k,I_k} S(I_k, D_k) + C_T(I_0 - D)^+$$

where $D = \sum_{l=0}^{n-1} D_l$. Denote $R_{i,j} = R(I_i)$ when $j = I_i$ for simplicity. We see $R_{i,0} = 0$ and $R_{n,j} = jC_T$.

We desire to optimize the expected revenue by maximizing expected remaining revenue at time i given all information from time 0 to i. This is the same as choosing $u_{i,j}$ at time i for all possible j, $1 \leq j \leq I_0$ to maximize

$$\mathbb{E}[R_{i,j}] = \sum_{k=i}^{n-1} u_{k,I_k} \mathbb{E}[S(I_k, D_k)] + C_T \mathbb{E}[(I_0 - D)^+] \tag{3.3}$$

subject to the state equation $I_{i+1} = I_i - S(I_i, D_i)$.

Using the law of total expectation we can write

$$\begin{aligned} \mathbb{E}[R_{i,j}] &= \mathbb{E}(\mathbb{E}[R_{i,j}|D_i]) \\ &= \sum_{k \geq 0} \mathbb{E}[R_{i,j}|D_i = k]\mathbb{P}(D_i = k). \end{aligned}$$

For $k \geq 2$ we have $\mathbb{P}(D_i = k) = o(\Delta t)$. Recall that the sum of little-o terms is still little-o to get

$$\mathbb{E}[R_{i,j}] = \mathbb{E}[R_{i,j}|D_i = 0]\mathbb{P}(D_i = 0) + \mathbb{E}[R_{i,j}|D_i = 1]\mathbb{P}(D_i = 1) + o(\Delta t). \tag{3.4}$$

Let $E_{i,j} = \mathbb{E}[R_{i,j}]$. If $j > 0$ then

$$\begin{aligned}
\mathbb{E}[R_{i,j}|D_i = 0] &= \mathbb{E}\left[\sum_{k=i}^{n-1} u_{k,I_k} S(I_k, D_k) + C_T(I_0 - D)^+\right] && (3.5)\\
&= u_{i,j}S(I_i, 0) + \mathbb{E}\left[\sum_{k=i+1}^{n-1} u_{k,I_k} S(I_k, D_k) + C_T(I_0 - D)^+\right] && (3.6)\\
&= E_{i+1,j}. && (3.7)
\end{aligned}$$

Similarly, it can be shown that

$$\mathbb{E}[R_{i,j}|D_i = 1] = u_{i,j} + E_{i+1,j-1}. \tag{3.8}$$

Substituting Equations 3.7 and 3.8 in we can solve for $\mathbb{E}_{i,j}$ in Equation 3.4,

$$E_{i,j} = (1 - \lambda_{i,j}\Delta t)E_{i+1,j} + \lambda_{i,j}\Delta t(u_{i,j} + E_{i+1,j-1}) + o(\Delta t). \tag{3.9}$$

We now have a dynamical program with boundary condition $R_{n,j} = jC_T$. We can back iterate to solve the program by solving for $u^*_{i,j}$ in the subproblem. We do this by taking the derivative of Equation 3.9 with respect to $u_{i,j}$ to get

$$0 = \lambda'_{i,j}(u^*_{i,j})\Delta t\left(u^*_{i,j} + E_{i+1,j-1} - E_{i+1,j}\right) + \lambda_{i,j}(u^*_{i,j})\Delta t. \tag{3.10}$$

We can now solve for $u^*_{i,j}$ to solve the subproblem. Given $u^*_{i,j}$ in any time period $i\Delta t$, we can back iterate to get the optimal pricing strategy. We can do this for different demand functions for $\lambda_{i,j}(u^*_{i,j})$. We will show the results for both the linear demand model and the

log-linear demand model.

3.1.1 Linear Demand. Suppose the Poisson parameter for demand is a linear function with respect to price. That is to say $\lambda_{i,j}(u^*_{i,j}) = a - bu^*_{i,j}$ for some $a, b > 0$ then 3.10 gives us

$$u^*_{i,j} = \frac{E_{i+1,j} - E_{i+1,j-1}}{2} + \frac{b}{2a}. \tag{3.11}$$

Figures 3.1 and 3.2 below show the optimal sales price strategy as well as the expected remaining revenue.

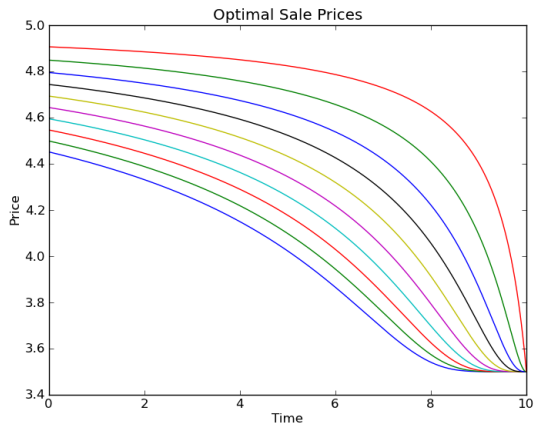

Figure 3.1: Optimal sale prices for each inventory level I. Assume $I_0 = 10$, $\lambda(u) = 10 - 2u$, $C_0 = 4$, $C_T = 2$, and $T = 10$.

Figure 3.1 shows the optimal pricing strategy measured in price over the time of good's lifespan. We see that as the expiration time approaches, the optimal price decreases as retailers are trying to "dump" their inventory. This is a common practice in inventory management. Figure 3.2 is a plot of the expected remaining revenue over time. We see that as time expires, the expected remaining revenue decreases. We see this decrease because the retailer has lowered prices in order to get as much revenue as possible before the expiration time. At expiration, the salvage price is lower than what the retailer could normally sell at so he tries to pull in as much revenue while it's still worth something.

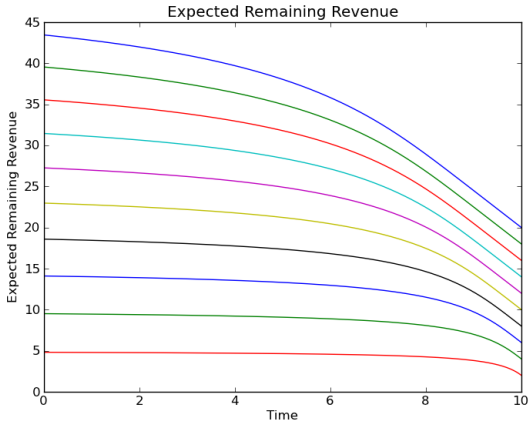

Figure 3.2: Expected remaining revenue for each inventory level I. Again assume $I_0 = 10$, $\lambda(u) = 10 - 2u$, $C_0 = 4$, $C_T = 2$, and $T = 10$.

3.1.2 Log-Linear Demand. Now suppose the Poisson parameter for demand is a log-linear function with respect to demand. In other words, $\lambda_{i,j}(u^*_{i,j}) = a(u^*_{i,j})^{-b}$ for some $b > 1$ then 3.10 gives us

$$u^*_{i,j} = \frac{b}{b-1}\left(E_{i+1,j} - E_{i+1,j-1}\right). \tag{3.12}$$

Graphs of the optimal pricing strategy and the expected remaining revenue will be similar to those in Figures 3.1 and 3.2.

3.2 Optimal Inventory

Solving the dynamic programming problem gives the expected remaining revenue of each level of inventory at any given time and a pricing strategy that will generate that expected remaining revenue. Clearly the more inventory, the higher the expected remaining revenue; a retailer, however, is concerned about maximizing profit, not just revenue. Knowing the expected remaining revenue will help the retailer choose which inventory to start out at to maximize profit. Recall that profit is given by total revenue minus total costs. Assume that the total cost is given by the inventory level, I_0, multiplied by the wholesale price, C_0, of the

good, $TC(I_0) = C_0 I_0$. In other words,

$$\Pi_{I_0} = \sum_{k=0}^{n-1} u_{k,I_k} S(I_k, D_k) + C_T(I_0 - \sum_{l=0}^{n-1} D_l)^+ - C_0 I_0.$$

Recall that E_{0,I_0} is the expected remaining revenue, $\mathbb{E}[R_{0,I_0}]$, at time $t = 0$ with I_0 remaining units of inventory. Then the expect profit can be found to be

$$\begin{aligned} \mathbb{E}[\Pi_{I_0}] &= \mathbb{E}\left[\sum_{k=0}^{n-1} u_{k,I_k} S(I_k, D_k) + C_T(I_0 - \sum_{l=0}^{n-1} D_l)^+\right] - C_0 I_0 \\ &= E_{0,I_0} - C_0 I_0. \end{aligned}$$

We can numerically find the optimal initial inventory level, I_0^*, by plotting the expected profit values at each inventory level in consideration. This is easy now that we already have the expected remaining revenue values, E_{0,I_0} at time $t = 0$, from the dynamic programming problem. Figure 3.3 show the expected profit across the different initial inventory levels. We see that in our previous example, there is a clear optimum value of inventory at $I_0 = 8$. We also see that the expected profit of the company with initial inventory 8, is slightly above 3.5.

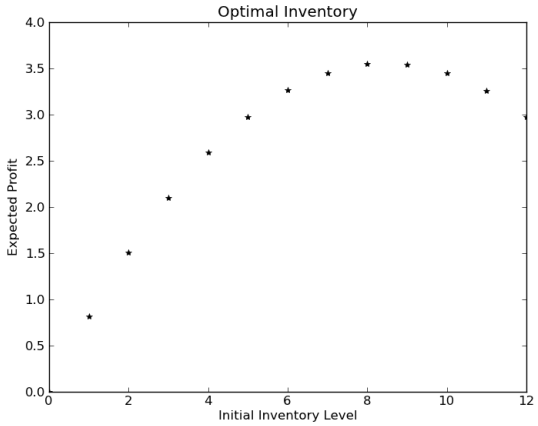

Figure 3.3: Optimal inventory in a one good market over initial inventory level I. Assume $\lambda(u) = 10 - 2u$, $C_0 = 4$, $C_T = 2$, and $T = 10$.

3.3 Inventory Probabilities

A retailer may be interested in finding the expected remaining inventory, $\mathbb{E}[I_T]$. One potential reason to calculate $\mathbb{E}[I_T]$ is to price a European inventory option. A European inventory option is similar to a European option on a stock. A European inventory option gives the owner the right to sell inventory at a set strike price, K, at a given time, T. The strike price is assumed to be above the salvage price, $K \geq C_T$, otherwise the option is useless. Although buying an inventory option doesn't increase the expected profit to the retailer, it has been proven to eliminate some risk through decreasing the variance of the expected profit. Note that $I_T = (I_0 - \sum_{l=0}^{n-1} D_l)^+$. Consider the profit function of a retailer without an inventory option

$$\begin{aligned} \Pi &= \sum_{k=0}^{n-1} u_{k,I_k} S(I_k, D_k) + C_T (I_0 - \sum_{l=0}^{n-1} D_l)^+ - C_0 I_0 \\ &= \sum_{k=0}^{n-1} u_{k,I_k} S(I_k, D_k) + C_T I_T - C_0 I_0 \end{aligned}$$

and with an inventory option

$$\hat{\Pi} = \sum_{k=0}^{n-1} u_{k,I_k} S(I_k, D_k) + K I_T - C_0 I_0 - p$$

where p is the cost of the option. The cost neutral price of the option is found when $\mathbb{E}[\Pi] = \mathbb{E}[\hat{\Pi}]$. It can be seen that this price is given by

$$p = (K - C_T)\mathbb{E}[I_T].$$

We note that with an inventory option, the optimal pricing strategy will likely change but it has been show that given a set pricing strategy, $\{u_{i,j}\}_{j=0}^{I_0}$, the expected profit with and

without the inventory option is the same [1]. In other words,

$$\mathbb{E}\left[\Pi|\{u^*_{i,j}\}_{j=0}^{I_0}\right] = \mathbb{E}\left[\hat{\Pi}|\{u^*_{i,j}\}_{j=0}^{I_0}\right].$$

Now let $P_j(t) = \mathbb{P}(I(t) = j)$ denote the probability that the inventory level is at j units at time t. Consider the probability distribution $\mathbf{P(t)} = (\mathbf{P_0(t)}, \mathbf{P_1(t)}, \ldots, \mathbf{P_{I_0}(t)})$ where I_0 is the initial inventory level. By definition of expected value, $\mathbb{E}[I_T] = \sum_{j=0}^{I_0} jP_j(T)$ and the price of the European option is

$$\begin{aligned} p &= (K - C_T)\mathbb{E}[I_T] \\ &= (K - C_T)\sum_{j=0}^{I_0} jP_j(T). \end{aligned}$$

Now we explain how to find $P_j(T)$. First discretize the time interval $[0, T]$ into n equal subintervals. So $\Delta t = T/n$ and denote $P_{i,j} = P_j(i\Delta t)$. Because the demand is a Poisson process we can use Equation 3.1 to see that the following transitional probabilities hold between probability states over time:

$$\begin{aligned} &\mathbb{P}(I_{i+1} = j|I_i = j) = 1 - \lambda_{i,j}\Delta t + o(\Delta t) \\ &\mathbb{P}(I_{i+1} = j|I_i = j+1) = \lambda_{i,j+1}\Delta t + o(\Delta t) \\ &\mathbb{P}(I_{i+1} = j|I_i \geq j+2) = o(\Delta t). \end{aligned}$$

By using laws of probability and partitioning the probability space, we also know that

$$\begin{aligned} P_{i+1,j} = \mathbb{P}(I_{i+1} = j) &= \sum_{k=0}^{I_0} \mathbb{P}(I_{i+1} = j|I_i = k)\mathbb{P}(I_i = k) \\ &= \sum_{k=0}^{I_0} \mathbb{P}(I_{i+1} = j|I_i = k)P_{i,k} \end{aligned}$$

We can simplify this by including the transition probabilities from previous inventory level as follows

$$P_{i+1,j} = \begin{cases} (1-\lambda_{i,I_0}\Delta t)P_{i,I_0} + o(\Delta t) & j = I_0 \\ (1-\lambda_{i,I_0}\Delta t)P_{i,j} + \lambda_{i,j+1}\Delta t P +_{i,j+1} + o(\Delta t) & 0 < j < I_0 \\ P_{i,0} + \lambda_{i,1}\Delta t P_{i,1} + o(\Delta t) & j = 0 \end{cases} . \tag{3.13}$$

Note that $\mathbf{P}(0) = (0, 0, \ldots, 0, 1)$ since at time $t = 0$, there is exactly I_0 units of inventory by definition. We can now use our optimal pricing strategy which we already found and our initial condition to find the probability distribution $\mathbf{P}(t)$ at any time t.

In Figure 3.4 we start with initial inventory level of $I_0 = 8$ since that was our optimal inventory level. The blue line with the first spike above the rest is when inventory is at the initial inventory level. As we progress to the following lines with spikes, we decrement the inventory by one every time, giving the last green line the inventory level of zero. Given the inventory probabilities at time T, we can calculate $\mathbb{E}[I_T] = 0.1103$ and $p = (K - C_T)\mathbb{E}[I_T] = 0.1103$.

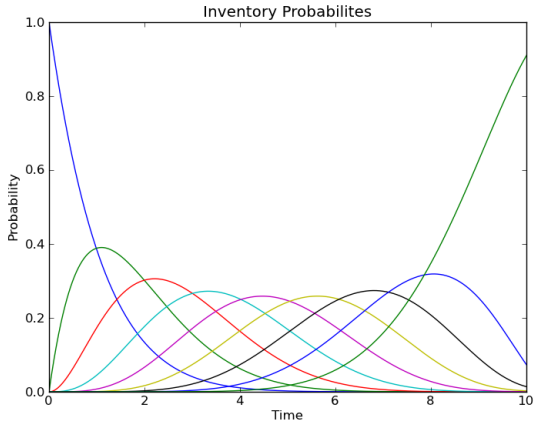

Figure 3.4: Inventory probabilities over time. Let $\lambda(u) = 10 - 2u$, $C_0 = 4$, $C_T = 1$, $K = 2$ and $T = 10$.

3.4 Wholesaler Optimization

A wholesaler selling the good to the distributor may notice that the firm is making a greater profit than they think the distributor warrants. The wholesaler may try to capture some of the profit by adjusting the wholesale price, C_0. We can numerically solve for the optimal wholesale price for the wholesaler by simply maximizing the wholesaler's profit function. Assume it's revenue is simply the optimal initial inventory for the firm, I_0^*, multiplied by the per unit profit, $(C_0 - C_T)$ as seen below.

$$\Pi_W = I_0^* * (C_0 - C_T)$$

We know that the optimal initial inventory, I_0^*, is a function of C_0 but we solved for I_0^* numerically; so there is no closed form solution. We can however, solve for the optimal wholesale price, C_0, numerically. In Figures 3.5 and 3.6 we graph the wholesaler's profit as a function of wholesale price. The sawtooth shape is caused because as we go right on the x-axis, there is a short period when the optimal inventory is the same but the price is increasing, resulting in the increasing part of the sawtooth. As the price continues to increase, the optimal inventory drops, resulting in the decreasing part of the sawtooth.

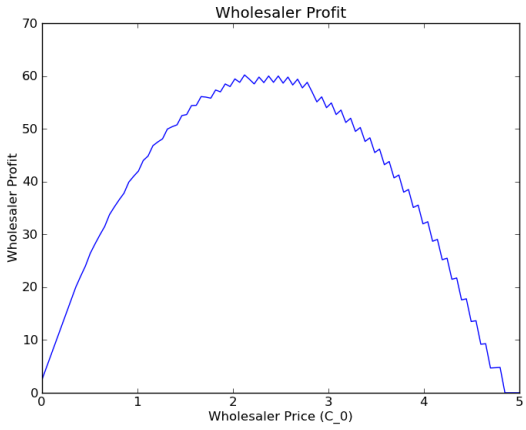

Figure 3.5: Optimal wholesale price given $\lambda(u) = 10 - 2u$, $C_T = 0$ and $T = 10$.

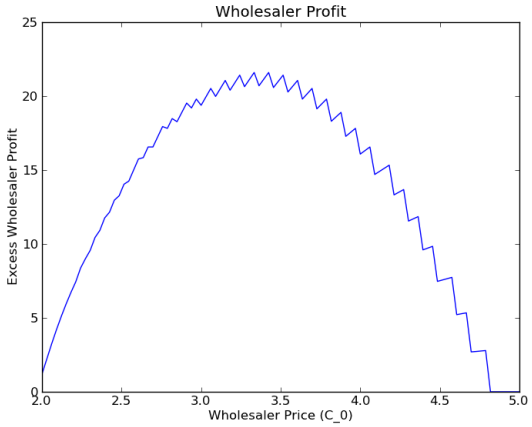

Figure 3.6: Optimal wholesale price given $\lambda(u) = 10 - 2u$, $C_T = 2$ and $T = 10$.

We note that if the wholesale price is below two, then the firm has an arbitrage opportunity where the buy price is less than the guaranteed sales price. We can eliminate the arbitrage opportunity in one of two ways. Either consider the market without a salvage price or consider the wholesale prices greater than two and also look at the revenues of all the goods after the first two units of price. Considering the market without a salvage price (as we did in Figure 3.5) will change the optimal pricing strategy for the firm and will ultimately change the dynamics of the whole problem. Considering the wholesale prices greater than two and looking at the revenues from all the goods after the first two units of price is equivalent to considering the marginal cost of the wholesaler equal to two. We do this because we're interesting in finding the optimum profit we can gain from the firm and not towards whoever is buying the salvage priced item. This way, there's no arbitrage opportunities and we successfully see what the optimal wholesale price is. In our case, we see that the optimal wholesale price is close to 3.5.

Chapter 4. Duopolistic Newsvendor Problem

A duopoly is an oligopoly with two firms. In other words, a duopoly is a market with only two goods sold by different firms. These goods can be homogeneous or heterogeneous goods.

Homogeneous goods refers to goods that are identical; whereas, heterogeneous goods are not the same goods but they sell in the same market. In the context of the newsvendor problem, we could consider two newspaper boys selling the same paper, the New York Times, versus two newspaper boys selling the New York Times and the Washington Post respectively as examples of homogeneous versus heterogeneous goods. Duopolistic models can also give us insights on market where two firms are the dominant players in a market.

Consider the demand of two goods in a duopolistic market as given by the random variables, X_a and X_b, which are Poisson processes. Assume the Poisson parameters are functions of the price of the two goods, $\lambda_a(u_a, u_b), \lambda_a(u_a, u_b)$ respectively. Firms a and b sell goods a and b respectively. Let the firms have an initial inventory amounts, $I_{a,0}, I_{b,0}$ respectively. Allow the firms to vary their prices and let $\{\{u_{a,j,k}(t)\}_{j=1}^{I_{a,0}}\}_{k=1}^{I_{b,0}}$ and $\{\{u_{b,j}(t)\}_{j=1}^{I_{a,0}}\}_{k=1}^{I_{b,0}}$ be the respective set of functions representing the firms pricing strategies given the respective inventory at time $t \in [0, T]$ with j, k remaining inventory units of goods a, b respectively. Note that the optimal price of good a depends on the inventory levels of both good a and b and we accordingly adapt our pricing strategies to incorporate this. At time T the goods have expired and are now worth some salvage prices $C_{a,T}$ and $C_{b,T}$ respectively. The remaining inventories for time t is given by $I_{a,t}, I_{b,t}$.

The cross price elasticity, $E_{a,b}$, is the percent change in quantity demanded for good a divided by the percent change in the price of good b. This can be written mathematically as

$$E_{a,b} = \frac{\partial Q_a}{\partial P_b} \frac{P_b}{Q_a}.$$

In our case, the demand is a random variable but we can use the Poisson parameter λ_a, which is the mean of the random variable, to represent our demand function, Q_a. We have

$$E_{a,b} = \frac{\partial \lambda_a}{\partial P_b} \frac{P_b}{\lambda_a}. \tag{4.1}$$

Two goods are said to be a complementary goods if the cross price elasticity is positive. If the cross price elasticity is negative, they are said to be substitutes. Since both P_b and λ_a are always nonnegative, the sign of the cross price elasticity is solely determined by $\frac{\partial \lambda_a}{\partial P_b}$. It is worth while to examine both cases as the behavior of the optimal sale price strategy changes depending on whether the goods are complements or substitutes. If $E_{a,b} = 0$ and $E_{b,a} = 0$, then we say the goods are neither substitutes nor complements.

Given a pricing strategy, we can find the demand functions for the Poisson rates and the cross price elasticity of the goods. So once we find that, we'll be able to calculate the Poisson rates, the cross price elasticities, and the expected remaining revenues of both goods.

4.1 Pricing Strategy

In order to find the optimal pricing strategy, discretize the time interval into time spans of length Δt. We assume that on the discretized time interval, the firm fixes the price. Similar to the newsvendor problem, let $S(I_{a,k}, D_{a,k})$ represent the amount of units of good a sold at time k as a function of remaining inventory $I_{a,k}$ and demand $D_{a,k}$. We let $u_{a,i,j,k}$ represent the sales price of good a at time $i\Delta t$ with remaining inventory levels j, k respectively for goods a, b. We write retailer a's profit function, Π_a as

$$\Pi_a = \sum_{k=0}^{n-1} u_{a,k,I_{a,k},I_{b,k}} S(I_{a,k}, D_{a,k}) + C_{a,T} I_{a,T} - C_{a,0} I_{a,0}$$

Take the expected value of Π_a we can find that

$$\mathbb{E}[\Pi_a] = \sum_{k=0}^{n-1} u_{a,k,I_{a,k},I_{b,k}} \mathbb{E}[S(I_{a,k}, D_{a,k})] + C_{a,T}\mathbb{E}[I_T] - C_{a,0}I_{a,0}$$

We've assumed that the prices $u_{a,i,j,k}, u_{b,i,j,k}$ are fixed on each small interval of time. Fixing the prices on small intervals implies that the Poisson parameters $\lambda_{a,i,j,k}, \lambda_{b,i,j,k}$ are also fixed on the time interval. We can then state that

$$\begin{aligned} \mathbb{P}(\quad X_a(t+\Delta t) \quad &= n+m | X_a(t) = n) \\ &= \begin{cases} 1 - \lambda_a(u_{a,j}(t))\Delta t + o(\Delta t) & \text{if } m = 0 \\ \lambda_a(u_{a,j}(t))\Delta t + o(\Delta t) & \text{if } m = 1 \\ o(\Delta t) & \text{if } m \geq 2 \end{cases} . \end{aligned}$$

Which gives us

$$D_{a,i} = \begin{cases} 0 & \text{w/ prob } 1 - \lambda_{a,i,j,k}\Delta t + o(\Delta t) \\ 1 & \text{w/ prob } \lambda_{a,i,j,k}\Delta t + o(\Delta t) \\ \geq 2 & \text{w/ prob } o(\Delta t) \end{cases} \tag{4.2}$$

for our demand function of good a on interval i. So the amount of inventory a sold is still represented by

$$S_a(I_{a,i}, D_{a,i}) = min\{I_{a,i}, D_{a,i}\}.$$

The same state equation,

$$I_{a,i+1} = I_{a,i} - S(I_{a,i}, D_{a,i}),$$

still holds.

Recall that after the initial inventory is set, maximizing total revenue is equivalent to maximizing total profit. As before, we will maximize the total revenue given set initial

inventory levels, $I_{a,0}, I_{b,0}$, and then discuss how to choose the optimal inventory in order to maximize profit. We can write total revenue for firm a as

$$R_a = \sum_{k=0}^{n-1} u_{a,k,I_{a,k},I_{b,k}} S(I_{a,k}, D_{a,k}) + C_{a,T} I_{a,T}$$

and remaining revenue as

$$R(I_{a,i}) = \sum_{k=i}^{n-1} u_{a,k,I_{a,k},I_{b,k}} S(I_{a,k}, D_{a,k}) + C_{a,T} I_{a,T}.$$

Note that the total revenue for firm b is equivalent up to symmetry.

Choose $u_{a,i,j,k}, u_{b,i,j,k}$ at time i for all $1 \leq j \leq I_{a,0}$, $1 \leq k \leq I_{b,0}$ to maximize their respective total expected revenue functions

$$\mathbb{E}[R_{a,i,j,k}] = \sum_{\ell=i}^{n-1} u_{a,i,j,k} \mathbb{E}[S_a(I_{a,\ell}, D_{a,\ell})] + C_{a,T} \mathbb{E}[I_{a,T}],$$

and $\mathbb{E}[R_{b,i,j,k}]$. From now on denote $\mathbb{E}[R_{a,i,j,k}]$ as $E_{a,i,j,k}$. By laws of total expectation we have

$$\begin{aligned}
E_{a,i,j,k} &= \mathbb{E}(\mathbb{E}[R_{a,i,j,k}|D_{a,i}, D_{b,i}]) \\
&= \sum_{c,d \geq 0} \mathbb{E}[R_{a,i,j,k}|D_{a,i} = c, D_{b,i} = d]\mathbb{P}(D_{a,i} = c, D_{b,i} = d) \\
&= \mathbb{E}[R_{a,i,j,k}|D_{a,i}, D_{b,i} = 0]\mathbb{P}(D_{a,i}, D_{b,i} = 0) \\
&\quad + \mathbb{E}[R_{a,i,j,k}|D_{a,i} = 0, D_{b,i} = 1]\mathbb{P}(D_{a,i} = 0, D_{b,i} = 1) \\
&\quad + \mathbb{E}[R_{a,i,j,k}|D_{a,i} = 1, D_{b,i} = 0]\mathbb{P}(D_{a,i} = 1, D_{b,i} = 0) \\
&\quad + \mathbb{E}[R_{a,i,j,k}|D_{a,i}, D_{b,i} = 1]\mathbb{P}(D_{a,i}, D_{b,i} = 1) \\
&\quad + o(\Delta t).
\end{aligned} \tag{4.3}$$

Examine each part of the summation and note that

$$\begin{aligned}
&\mathbb{E}[R_{a,i,j,k}|D_{a,i}, D_{b,i} = 0] \\
&\quad = \quad \mathbb{E}\left[\textstyle\sum_{\ell=i}^{n-1} u_{a,\ell,I_{a,\ell}} S_{a,\ell} + C_{a,T}(I_{a,0} - D_a)^+ | D_{a,i}, D_{b,i} = 0\right] \\
&\quad = \quad u_{a,i,j,k} S_{a,i}(I_{a,i}, 0) + \mathbb{E}\left[\textstyle\sum_{\ell=i+1}^{n-1} u_{a,\ell,I_{a,\ell}} S_{a,\ell} + C_{a,T}(I_{a,0} - D_a)^+\right] \\
&\quad = \quad E_{a,i+1,j,k}. \qquad (4.4)
\end{aligned}$$

Similarly, it can be show that

$$\begin{aligned}
\mathbb{E}[R_{a,i,j,k}|D_{a,i} = 0, D_{b,i} = 1] &= E_{a,i+1,j,k-1} && (4.5)\\
\mathbb{E}[R_{a,i,j,k}|D_{a,i} = 1, D_{b,i} = 0] &= u_{a,i,j,k} + E_{a,i+1,j-1,k} && (4.6)\\
\mathbb{E}[R_{a,i,j,k}|D_{a,i} = 1, D_{b,i} = 1] &= u_{a,i,j,k} + E_{a,i+1,j-1,k-1}. && (4.7)
\end{aligned}$$

We also note that assuming the Poisson rates are independent, we get

$$\begin{aligned}
\mathbb{P}(D_{a,i}, D_{b,i} = 0) &= [1 - \lambda_{a,i,j,k}\Delta t + o(\Delta t)] * [1 - \lambda_{b,i,j,k}\Delta t + o(\Delta t)] && (4.8)\\
\mathbb{P}(D_{a,i} = 0, D_{b,i} = 1) &= [1 - \lambda_{a,i,j,k}\Delta t + o(\Delta t)] * \lambda_{b,i,j,k}\Delta t && (4.9)\\
\mathbb{P}(D_{a,i} = 1, D_{b,i} = 0) &= \lambda_{a,i,j,k}\Delta t * [1 - \lambda_{b,i,j,k}\Delta t + o(\Delta t)] && (4.10)\\
\mathbb{P}(D_{a,i}, D_{b,i} = 1) &= \lambda_{a,i,j,k}\Delta t * \lambda_{b,i,j,k}\Delta t. && (4.11)
\end{aligned}$$

Substituting equations 4.4-4.11 in for the total expectation in equation 4.3 above, we get

$$\begin{aligned}
E_{a,i,j,k} &= [1 - \lambda_{a,i,j,k}\Delta t] * [1 - \lambda_{b,i,j,k}\Delta t]\,(E_{a,i+1,j,k}) \\
&\quad + [1 - \lambda_{a,i,j,k}\Delta t] * \lambda_{b,i,j,k}\Delta t * (E_{a,i+1,j,k-1}) \\
&\quad + \lambda_{a,i,j,k}\Delta t * [1 - \lambda_{b,i,j,k}\Delta t]\,(u_{a,i,j,k} + E_{a,i+1,j-1,k}) \\
&\quad + \lambda_{a,i,j,k}\Delta t * \lambda_{b,i,j,k}\Delta t\,(u_{a,i,j,k} + E_{a,i+1,j-1,k-1}) \\
&\quad + o(\Delta t)
\end{aligned}$$

We know that $R_{a,n,j,k} = jC_{a,T}$ where n is the number of discrete time intervals, $\left(\frac{T}{\Delta t}\right)$, because $C_{a,T}$ is the salvage price. We can now find $E_{a,i,j,k}$ and $E_{b,i,j,k}$ for all i, j, k by using a dynamical program to iterate backwards from this initial condition. We must also note that if j or k is zero, then the Poisson rate to sell the said good is zero. In other words, $\lambda_{a,i,0,k} = 0$ and $\lambda_{b,i,j,0} = 0$. So we know that

$$E_{a,i,j,0} = [1 - \lambda_{a,i,j,0}\Delta t] * (E_{a,i+1,j,0}) + \lambda_{a,i,j,0}\Delta t * (u_{a,i,j,0} + E_{a,i+1,j-1,0}) + o(\Delta t)$$

To find the optimal price for inventory levels j, k at time i, we take the derivative of the previous equations with respect to $u_{a,i,j,k}$, set them equal to zero, take the limit as Δt approaches zero and solve for $u^*_{a,i,j,k}$. Through symmetry we can infer that $u^*_{b,i,j,k}$ will be similar to $u^*_{a,i,j,k}$.

We find,

$$0 = \frac{\partial E_{a,i,j,0}}{\partial u_{a,i,j,0}} = -\lambda'_{a,i,j,0}\Delta t * (E_{a,i+1,j,0}) + \lambda'_{a,i,j,0}\Delta t * (u_{a,i,j,0} + E_{a,i+1,j-1,0}) + \lambda_{a,i,j,0}\Delta t$$

when $k = 0$ and

$$\begin{aligned}
0 = \frac{\partial E_{a,i,j,k}}{\partial u_{a,i,j,k}} &= -\lambda'_{a,i,j,k}\Delta t\left[1-\lambda_{b,i,j,k}\Delta t\right]\left(E_{a,i+1,j,k}\right) \\
&\quad -\left[1-\lambda_{a,i,j,k}\Delta t\right]\lambda'_{b,i,j,k}\Delta t\left(E_{a,i+1,j,k}\right) \\
&\quad -\lambda'_{a,i,j,k}\Delta t * \lambda_{b,i,j,k}\Delta t * \left(E_{a,i+1,j,k-1}\right) \\
&\quad +\left[1-\lambda_{a,i,j,k}\Delta t\right] * \lambda'_{b,i,j,k}\Delta t * \left(E_{a,i+1,j,k-1}\right) \\
&\quad +\lambda'_{a,i,j,k}\Delta t * \left[1-\lambda_{b,i,j,k}\Delta t\right]\left(u_{a,i,j,k}+E_{a,i+1,j-1,k}\right) \\
&\quad -\lambda_{a,i,j,k}\Delta t * \lambda'_{b,i,j,k}\Delta t\left(u_{a,i,j,k}+E_{a,i+1,j-1,k}\right) \\
&\quad +\lambda_{a,i,j,k}\Delta t * \left[1-\lambda_{b,i,j,k}\Delta t\right] \\
&\quad +\lambda'_{a,i,j,k}\Delta t * \lambda_{b,i,j,k}\Delta t\left(u^*_{a,i,j,k}+E_{a,i+1,j-1,k-1}\right) \\
&\quad +\lambda_{a,i,j,k}\Delta t * \lambda'_{b,i,j,k}\Delta t\left(u^*_{a,i,j,k}+E_{a,i+1,j-1,k-1}\right) \\
&\quad +\lambda_{a,i,j,k}\Delta t * \lambda_{b,i,j,k}\Delta t.
\end{aligned}$$

when $k \neq 0$. We can divide by Δt first and then take the limit as Δt approaches zero to find simplify down to

$$0 = -\lambda'_{a,i,j,0} * \left(E_{a,i+1,j,0}\right) + \lambda'_{a,i,j,0} * \left(u_{a,i,j,0}+E_{a,i+1,j-1,0}\right) + \lambda_{a,i,j,0} \tag{4.12}$$

when $k = 0$ and

$$0 = -\lambda'_{a,i,j,k}\left(E_{a,i+1,j,k}\right) - \lambda'_{b,i,j,k}\left(E_{a,i+1,j,k}\right) + \lambda'_{b,i,j,k} * \left(E_{a,i+1,j,k-1}\right) + \lambda'_{a,i,j,k} * \left(u_{a,i,j,k}+E_{a,i+1,j-1,k}\right) + \lambda_{a,i,j,k} \tag{4.13}$$

when $k \neq 0$.

We can now plot the optimal pricing strategy for any given demand function representing the Poisson parameter. The pricing strategy is given regardless if the products are complements or substitutes.

4.1.1 Linear Demand. For the rest of the paper, we will assume the Poisson parameters are linear functions with respect to the prices. In other words, the Poisson parameters are given by the system of equations

$$\begin{bmatrix} \lambda_1(t) \\ \lambda_2(t) \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} - \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix}$$

where $b_{11}, b_{22} > 0$. We make this assumption for simplicity and note that this is a common way to denote the demand for two goods [19]. It is assumed that $b_{11} > b_{12}$ and $b_{22} > b_{21}$, otherwise it is theoretically possible to get infinite demand.

We now find the optimal pricing strategy for the linear demand case. We can see that $\lambda'_{a,i,j,0} = \frac{\partial \lambda'_{a,i,j,k}}{\partial u_{a,i,j,k}} = -b_{11}$. Substituting $\lambda_{a,i,j,0}$ and $\lambda'_{a,i,j,0}$ into Equation 4.12, we get

$$\begin{aligned} 0 &= b_{11}\left(E_{a,i+1,j,0}\right) - b_{11}\left(u^*_{a,i,j,0} + E_{a,i+1,j-1,0}\right) + a_1 - b_{11}u^*_{a,i,j,0} - b_{12}u_{b,i,j,0} \\ u^*_{a,i,j,0} &= \frac{1}{2b_{11}}\left(a_1 + b_{11}(E_{a,i+1,j,0} - E_{a,i+1,j-1,0}) - b_{12}u_{b,i,j,0}\right) \end{aligned}$$

Similarly

$$u^*_{b,i,0,k} = \frac{1}{2b_{22}}\left(a_2 + b_{22}(E_{b,i+1,0,k} - E_{b,i+1,0,k-1}) - b_{21}u_{a,i,0,k}\right).$$

Now substitute $\lambda_{a,i,j,0}$ and $\lambda'_{a,i,j,0}$ into Equation 4.13 to get

$$\begin{aligned} 0 &= b_{11}\left(E_{a,i+1,j,k}\right) + b_{22}\left(E_{a,i+1,j,k}\right) - b_{22}\left(E_{a,i+1,j,k-1}\right) - b_{11}\left(u^*_{a,i,j,k} + E_{a,i+1,j-1,k}\right) \\ &\quad + (a_1 - b_{11}u^*_{a,i,j,k} - b_{12}u_{b,i,j,k}) \\ u^*_{a,i,j,k} &= \frac{1}{2b_{11}}\left(a_1 + b_{11}E_{a,i+1,j,k} + b_{21}E_{a,i+1,j,k} - b_{21}E_{a,i+1,j,k-1} - b_{11}E_{a,i+1,j-1,k} - b_{12}u_{b,i,j,k}\right) \end{aligned}$$

By symmetry we can find a similar optimal value for $u^*_{b,i,j,k}$. Set up the system of equations

and solve for $u^*_{a,i,j,k}$ and $u^*_{b,i,j,k}$ to get

$$\begin{bmatrix} u^*_{a,i,j,k} \\ u^*_{b,i,j,k} \end{bmatrix} = \begin{bmatrix} 2b_{11}b_{22} & b_{12}b_{22} \\ b_{21}b_{11} & 2b_{11}b_{22} \end{bmatrix}^{-1} \begin{bmatrix} b_{22}\left(a_1 + (b_{11}+b_{21})\,E_{a,i+1,j,k} - b_{21}E_{a,i+1,j,k-1} - b_{11}E_{a,i+1,j-1,k}\right) \\ b_{11}\left(a_2 + (b_{22}+b_{12})\,E_{b,i+1,j,k} - b_{12}E_{b,i+1,j-1,k} - b_{22}E_{b,i+1,j,k-1}\right) \end{bmatrix}$$

whenever $k \neq 0$.

The Figures 4.1 and 4.3 are the graphs of the optimal pricing strategies of a two good market with linear demand functions. The first graph is when the goods are substitutes and the second graph shows complementary goods. Similarly, the Figures 4.2 and 4.4 are the corresponding graphs of the expected remaining revenue.

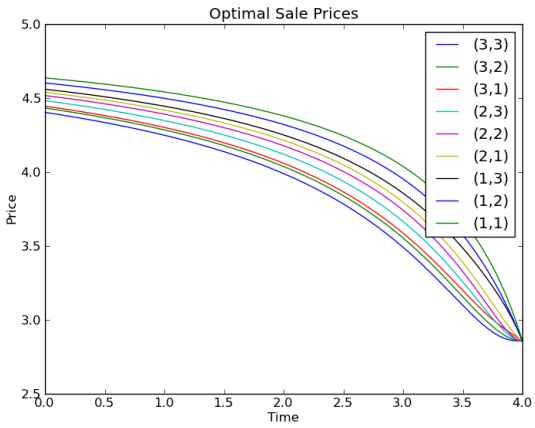

Figure 4.1: Optimal sale prices for two substitutes in a duopolistic market. Assume $I_{a,0} = I_{b,0} = 3$ and $\lambda_1(p_1, p_2) = 10 - 5p_1 + 3p_2$ and $\lambda_2(p_1, p_2) = 10 + 3p_1 - 5p_2$. Let $C_{a,0} = C_{b,0} = 4$, $C_{a,T} = C_{a,T} = 2$, and $T = 4$.

In both cases (substitutes and complements), a general rule of thumb would say that the remaining time till expiration is correlated with the sale prices and as the expiration time approaches, the sale price decreases. However, this isn't always true for the complementary goods. This alone is a phenomenon worth noting. Before we explain this phenomenon, we also note that some of the optimal sale price lines cross. This is due to the imbalance of inventory levels. At the beginning of the time period, the competitor's inventory level is less relevant as it is closer to the expiration date. This causes the sale prices to cross as time

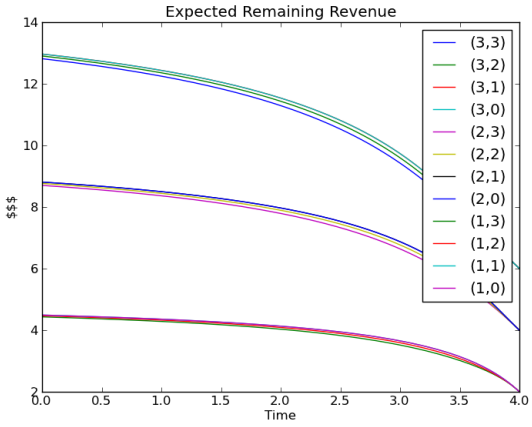

Figure 4.2: Expected remaining revenue for substitutes in a duopolistic market. Assume $I_{a,0} = I_{b,0} = 3$, $\lambda_1(p_1, p_2) = 10 - 5p_1 + 3p_2$, $\lambda_2(p_1, p_2) = 10 + 3p_1 - 5p_2$, $C_{a,0} = C_{b,0} = 4$, $C_{a,T} = C_{a,T} = 2$, and $T = 4$.

approaches its expiration.

Why does the behavior of the pricing strategy of the complementary goods differ from that of the substitutes? It is interesting to see that in some cases the optimal sale price in the complementary goods has a dip and is actually lower before its optimal sale price at the expiration time. Even though the sales price never goes below the salvage price, this behavior is still hard to explain; especially since it doesn't happen in the substitute case.

We do notice that this behavior happens when the firm has a lot of their product but the competing firm has low inventory levels. For example, this happens in Figure 4.3 when inventory levels are $(3, 1)$. The low inventory level firm has lots of time relatively to sell their few units of product. The firm wants to take advantage of the time so they raise their prices in order to increase the expected remaining revenue. This rise in price decreases the demand not only for the corresponding good but also the other good because they're complements. Since the higher inventory level firm has more inventory, relatively little time, and now a lower demand, they lower their prices in order to compensate. The low inventory firm keeps prices high until right before the expiration time. So as the expiration time gets closer, the

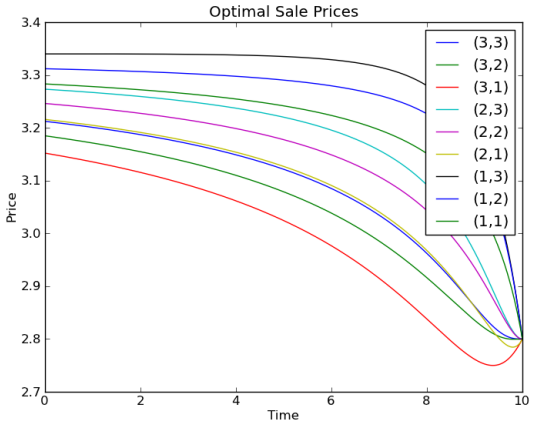

Figure 4.3: Optimal sale prices for complementary goods in a duopolistic market. Assume $I_{a,0} = I_{b,0} = 3$, $\lambda_1(p_1, p_2) = 10 - 2p_1 - p_2$, $\lambda_2(p_1, p_2) = 10 - p_1 - 2p_2$. Let $C_{a,0} = C_{b,0} = 3$, $C_{a,T} = C_{a,T} = 2$, and $T = 10$.

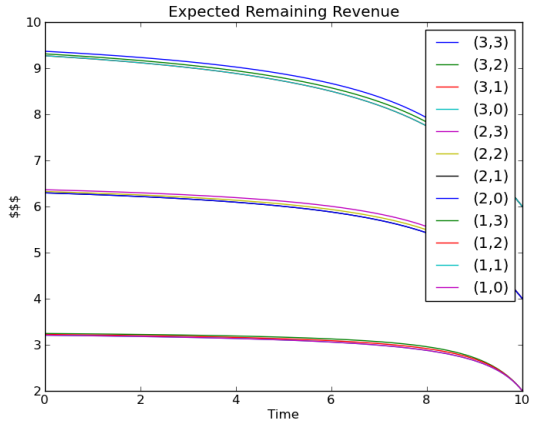

Figure 4.4: Expected remaining revenue for complementary goods in a duopolistic market. Assume $I_{a,0} = I_{b,0} = 3$, $\lambda_1(p_1, p_2) = 10 - 2p_1 - p_2$, $\lambda_2(p_1, p_2) = 10 - p_1 - 2p_2$, $C_{a,0} = C_{b,0} = 3$, $C_{a,T} = C_{a,T} = 2$, and $T = 10$.

low inventory level firm dramatically drops their prices which sends the demand for the other firm up due to the higher demand. This is an interesting phenomenon that isn't intuitive and captured by this model of demand.

4.2 Cross Price Elasticity

We determined that two goods are complements or substitutes if the cross price elasticity is positive or negative respectively and that the sign of the cross price elasticity is determined

by $\frac{\partial \lambda_a}{\partial P_b}$. It is easy to see that $\frac{\partial \lambda_a}{\partial P_b} = -b_{12}$. So the sign of the cross price elasticity is determined solely by the sign of b_{12}. Similarly the sign of $E_{b,a}$ is determined by b_{21}. Therefore, if b_{12} and b_{21} are positive, the goods are complements and if b_{12} and b_{21} are negative, the goods are substitutes.

Recall that the cross price elasticity of good a with respect to good b is given by

$$E_{a,b} = \frac{\partial \lambda_a}{\partial P_b} \frac{P_b}{\lambda_a}.$$

However the price and the Poisson rate are both functions of time so we should really write

$$E_{a,b}(t) = \frac{\partial \lambda_a}{\partial P_b} \frac{P_b(t)}{\lambda_a(t)}.$$

We calculated the optimal sales price strategies by discretizing the time intervals. Having calculated the optimal pricing strategies for both firms, we can find the corresponding Poisson rates and cross price elasticity for each time interval in the discretization by plugging in the optimal sale prices.

We write $E_{a,i,j,k}$ to denote the cross price elasticity of good a with respect to good b, at time $i\Delta t$ when there is still (j,k) levels of inventory for a and b respectively. We write

$$E_{a,i,j,k} = \frac{\partial \lambda_a}{\partial P_b} \frac{u^*_{b,i,j,k}}{\lambda_{a,i,j,k}} \tag{4.14}$$

where $\lambda_{a,i,j,k}$ is a function of the optimal sale prices $u^*_{a,i,j,k}$ and $u^*_{b,i,j,k}$. So although $\frac{\partial \lambda_a}{\partial P_b}$ is a constant in the linear demand case, the cross price elasticity $E_{a,i,j,k}$ is still a function of time and will change as the sales price and demand change. Figures 4.5 and 4.6 are the plots of the cross price elasticity as time progresses for each corresponding inventory level for markets with the substitutes and complementary goods respectively.

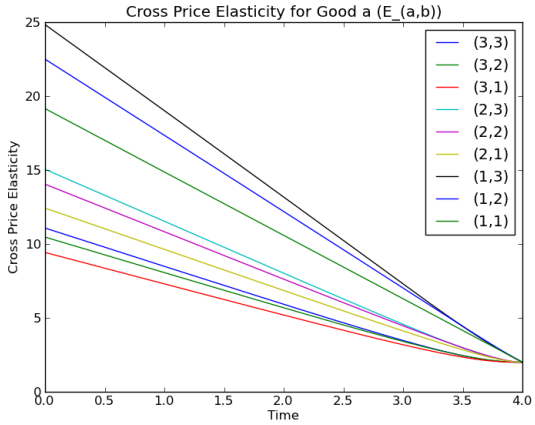

Figure 4.5: Cross price elasticity of a duopolistic market with homogeneous substitutes over time. Assume $I_{a,0} = I_{b,0} = 3$, $\lambda_1(p_1, p_2) = 10 - 5p_1 + 3p_2$, $\lambda_2(p_1, p_2) = 10 + 3p_1 - 5p_2$, $C_{a,0} = C_{b,0} = 4$, $C_{a,T} = C_{a,T} = 2$, and $T = 4$.

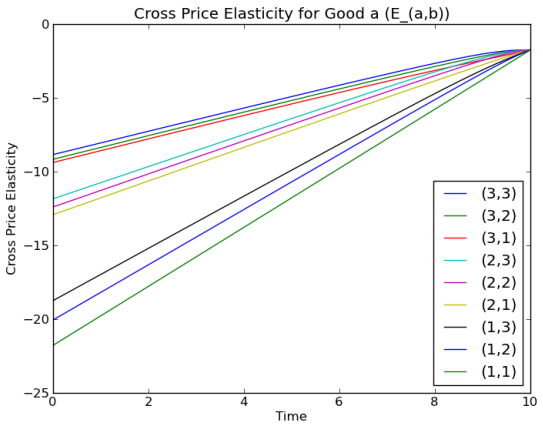

Figure 4.6: Cross price elasticity of a duopolistic market with homogeneous, complementary goods over time. Assume $I_{a,0} = I_{b,0} = 3$, $\lambda_1(p_1, p_2) = 10 - 2p_1 - p_2$, $\lambda_2(p_1, p_2) = 10 - p_1 - 2p_2$, $C_{a,0} = C_{b,0} = 3$, $C_{a,T} = C_{a,T} = 2$, and $T = 10$.

At the beginning of the time period, the magnitude of the elasticities are higher than at anywhere else. This is due because the prices are at their highest peak and the demand parameters, λ, are at the lowest at the beginning of the time period. Looking at Equation 4.14, it is easy to see why this would cause the magnitude of the cross price elasticity to be high.

Similarly, we see that the highest magnitudes of cross price elasticity for both the substitute and complementary goods occur when the respective good has the lowest inventory level. We explain this because the less inventory you have, the more likely you are to have high prices and low demand because you're trying to milk the system for as much as you can. Because your demand is low, the magnitude of the cross price elasticity is high.

Next we examine the effect of the competitor's inventory level on cross price elasticity. In the substitutes good market, the magnitudes of cross price elasticity are ordered according to the number of competitor's inventory. That is to say, the lower the competitor's inventory level correspond with lower the cross price elasticity holding all else constant. The complementary goods are opposite and the competitor's inventory level and the cross price elasticity are inversely proportional.

Although we would expect the competitors' inventory levels to influence cross price elasticity differently for substitutes and complements, we now have a way to measure their effects.

4.3 Optimal Inventory

Again we desire to find the optimal inventory level that maximizes profits for firm a. Recall that the total profit of firm a, assuming total cost, TC, has no overhead cost and is just given by $TC = C_{a,0}I_{a,0}$, is given by

$$\Pi_a = \sum_{k=0}^{n-1} u_{a,k,I_{a,k},I_{b,k}} S(I_{a,k}, D_{a,k}) + C_{a,T}I_{a,T} - C_{a,0}I_{a,0}$$

and the total revenue of firm a is given by

$$R_a = \sum_{k=0}^{n-1} u_{a,k,I_{a,k},I_{b,k}} S(I_{a,k}, D_{a,k}) + C_{a,T}I_{a,T}.$$

We can find the total profit by taking the difference between total revenue and the total cost as follows:

$$\Pi_a = R_a - C_{a,0} I_{a,0}. \tag{4.15}$$

To find the optimal inventory level, we desire to maximize the expect profit $\mathbb{E}[\Pi_a]$ with respect to $I_{a,0}$. We can take the expected value of each side of the profit function in Equation 4.15 to get

$$\mathbb{E}[\Pi_a] = \mathbb{E}[R_a] - C_{a,0} I_{a,0}. \tag{4.16}$$

We already have the expected revenue given a fixed inventory level from the previous section so we can easily find the expected profit by subtracting the wholesaler price, $C_{a,0}$, times the given initial inventory level, $I_{a,0}$. There is no closed form for the optimal inventory, but we can numerically find it by plotting the expected profits over the initial inventory levels. Figures 4.7 and 4.8 are plot of the expected profits over the initial inventory levels for product a. They can be used to find the Nash Equilibrium, if it exists, in the market.

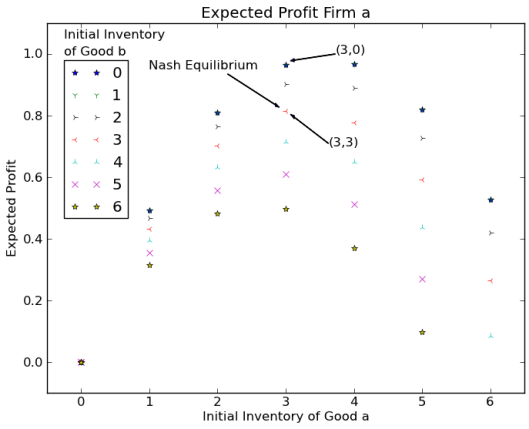

Figure 4.7: Optimal inventory in a duopolistic market with substitutes. Assume $\lambda_1(p_1, p_2) = 10 - 5p_1 + 3p_2$, $\lambda_2(p_1, p_2) = 10 + 3p_1 - 5p_2$, $C_{a,0} = C_{b,0} = 4$, $C_{a,T} = C_{a,T} = 2$, and $T = 4$.

At first glance, one may think that the optimal inventory level is three, in Figure 4.7,

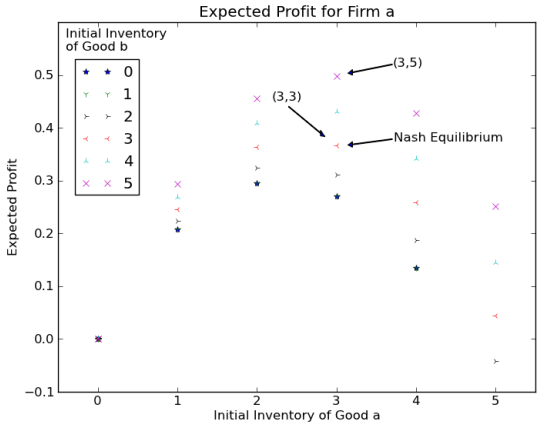

Figure 4.8: Optimal inventory in a duopolistic market with complementary goods. Assume $\lambda_1(p_1, p_2) = 10 - 2p_1 - p_2$, $\lambda_2(p_1, p_2) = 10 - p_1 - 2p_2$, $C_{a,0} = C_{b,0} = 3$, $C_{a,T} = C_{a,T} = 2$, and $T = 10$.

$I_{a,0}$=3 units, and occurs when firm b starts with $I_{b,0} = 0$. However, by symmetry, we know that if firm a chooses $I_{a,0} = 3$, then firm b will choose an inventory level of $I_{b,0} = 3$. At this value, there is no reason for either company to deviate, so we know that this, $(I_{a,0}, I_{b,0}) = (3, 3)$, is a Nash Equilibrium.

Similar to the substitute case, we can choose arbitrary starting values for initial inventory $(I_{a,0}, I_{b,0})$ and work our way to the Nash Equilibrium. Let's start at $(I_{a,0}, I_{b,0}) = (3, 5)$. By symmetry firm b would move initial inventory from $I_{b,0} = 5$ to $I_{b,0} = 3$ where neither company has incentive to deviate. Therefore the Nash Equilibrium is $I_{a,0} = I_{b,0} = 3$.

Technically the complementary good case has two Nash Equilibria found at $(I_{a,0}, I_{b,0}) = (3, 3)$ and $(I_{a,0}, I_{b,0}) = (2, 2)$. But we can assume that both firms will see that the other firm can make more profit if they both start with three units of inventory and will consequently move to that equilibrium. However, this is the problem with such an empirical analysis; an optimal initial inventory level may not exist. One of the reasons this happens is because the initial inventory levels are discrete values and we can not start with fractions of an inventory unit.

We should also note that when the goods are substitutes, the firms do better when their opponents have less inventory. On the other hand, when the goods are complements, the firms do better when their opponents have more inventory. This is because when a firm has low inventory, they have higher prices. So in the substitute case, this will increase the demand for the other good. However, in the complement case, higher opponent prices will cause a decrease in demand.

4.4 Inventory Probabilities

Again we may wish to find the expected remaining inventory, $\mathbb{E}[I_T]$ in order to price an inventory option. If we had an inventory option on good a where the salvage price is $C_{a,T}$ and the strike price is K_a. Then we should price the option as follows

$$p_a = (K_a - C_{a,T})\mathbb{E}[I_{a,T}]$$

where $I_{a,T}$ is the remaining inventory at time T, the expiration date.

We see that the original profit function was given by total revenue minus the total cost as shown below:

$$\begin{aligned}
\Pi &= \sum_{k=0}^{n-1} u_{a,k,I_{a,k},I_{b,k}} S(I_{a,k}, D_{a,k}) + C_{a,T}\left(I_{a,0} - \sum_{l=0}^{n-1} D_{a,l}\right)^+ - C_{a,0} I_{a,0} \\
&= \sum_{k=0}^{n-1} u_{a,k,I_{a,k},I_{b,k}} S(I_{a,k}, D_{a,k}) + C_{a,T} I_{a,T} - C_{a,0} I_{a,0} \\
&= \left(\sum_{k=0}^{n-1} (u_{a,k,I_{a,k},I_{b,k}} - C_{a,0}) S(I_{a,k}, D_{a,k})\right) + (C_{a,T} - C_{a,0})\, I_{a,T}
\end{aligned}$$

So the expected profit is given by

$$\mathbb{E}[\Pi] = \left(\sum_{k=0}^{n-1} (u_{a,k,I_{a,k},I_{b,k}} - C_{a,0}) \mathbb{E}[S(I_{a,k}, D_{a,k})]\right) + (C_{a,T} - C_{a,0})\, \mathbb{E}[I_{a,T}]$$

Similarly with the inventory option we get

$$\begin{aligned}
\hat{\Pi} &= \sum_{k=0}^{n-1} u_{a,k,I_{a,k},I_{b,k}} S(I_{a,k}, D_{a,k}) + K_a I_{a,T} - C_{a,0} I_{a,0} - (K_a - C_{a,T})\mathbb{E}[I_{a,T}] \\
&= \left(\sum_{k=0}^{n-1} (u_{a,k,I_{a,k},I_{b,k}} - C_{a,0}) S(I_{a,k}, D_{a,k}) \right) + (K_a - C_{a,0})\, I_{a,T} - (K_a - C_{a,T})\mathbb{E}[I_{a,T}] \\
\mathbb{E}[\hat{\Pi}] &= \left(\sum_{k=0}^{n-1} (u_{a,k,I_{a,k},I_{b,k}} - C_{a,0}) \mathbb{E}[S(I_{a,k}, D_{a,k})] \right) + (K_a - C_{a,0})\, \mathbb{E}[I_{a,T}] - (K_a - C_{a,T})\mathbb{E}[I_{a,T}] \\
&= \left(\sum_{k=0}^{n-1} (u_{a,k,I_{a,k},I_{b,k}} - C_{a,0}) \mathbb{E}[S(I_{a,k}, D_{a,k})] \right) + (C_{a,T} - C_{a,0})\, \mathbb{E}[I_{a,T}]
\end{aligned}$$

Fix a pricing strategy. Call it $\mathbf{u}^*$. It can be shown that the expected profits with and without the inventory are the same, $\mathbb{E}[\Pi|\mathbf{u}^*] = \mathbb{E}[\hat{\mathbf{\Pi}}|\mathbf{u}^*]$, by substituting the pricing strategy into the expected profit functions.

In order to find the price of the option, we must find $\mathbb{E}[I_T]$. To do this, discretize the time interval into n equal intervals so that $T = n * \Delta t$. We can calculate $\mathbb{E}[I_{a,T}]$ using the probabilities that we're in a given state as calculated in the previous section. We have

$$\mathbb{E}[I_{a,T}] = \sum_{j=0}^{I_{a,0}} j \mathbb{P}(I_{a,T} = j).$$

Similarly to the one good newsvendor problem, we consider the probability distribution as a matrix where the rows indicate the inventory level for product a and the columns represent the inventory level for product b. We can write this probability distribution matrix as

$$\mathbf{P(t)} = \begin{pmatrix} P_{0,0}(t) & P_{0,1}(t) & \dots & P_{0,I_{b,0}}(t) \\ P_{1,0}(t) & P_{1,1}(t) & \dots & P_{1,I_{b,0}}(t) \\ \vdots & \vdots & \ddots & \vdots \\ P_{I_{a,0},0}(t) & P_{I_{a,0},1}(t) & \dots & P_{I_{a,0},I_{b,0}}(t) \end{pmatrix}.$$

We can see that $\mathbb{P}(I_{a,T} = j) = \sum_{k=0}^{I_{b,0}} P_{n,j,k}$ where $P_{n,j,k}$ is the corresponding value in the probability matrix $\mathbf{P_{j,k}(n)}$. So we know

$$\mathbb{E}[I_{a,T}] = \sum_{j=0}^{I_{a,0}} j \sum_{k=0}^{I_{b,0}} P_{n,j,k}. \tag{4.17}$$

With this we can find the break even price for the inventory option at price

$$p_a = (K_a - C_{a,T})\mathbb{E}[I_{a,T}] = (K_a - C_{a,T}) \sum_{j=0}^{I_{a,0}} j \sum_{k=0}^{I_{b,0}} P_{n,j,k}. \tag{4.18}$$

Since we have assumed that demand is a Poisson process, we can calculate $P_{n,j,k}$ using the following transitional probabilities associated with state progression over time:

$$\mathbb{P}(I_{a,i+1} = j, I_{b,i+1} = k | I_{a,i} = j, I_{b,i} = k) = [1 - \lambda_{a,i,j,k}\Delta t + o(\Delta t)] * [1 - \lambda_{b,i,j,k}\Delta t + o(\Delta t)]$$

$$\mathbb{P}(I_{a,i+1} = j, I_{b,i+1} = k | I_{a,i} = j, I_{b,i} = k+1) = [1 - \lambda_{a,i,j,k}\Delta t + o(\Delta t)] * [\lambda_{b,i,j,k}\Delta t + o(\Delta t)]$$

$$\mathbb{P}(I_{a,i+1} = j, I_{b,i+1} = k | I_{a,i} = j+1, I_{b,i} = k) = [\lambda_{a,i,j,k}\Delta t + o(\Delta t)] * [1 - \lambda_{b,i,j,k}\Delta t + o(\Delta t)]$$

$$\mathbb{P}(I_{a,i+1} = j, I_{b,i+1} = k | I_{a,i} = j+1, I_{b,i} = k+1) = [\lambda_{a,i,j,k}\Delta t + o(\Delta t)] * [\lambda_{b,i,j,k}\Delta t + o(\Delta t)]$$

$$\mathbb{P}(I_{a,i+1} = j | I_{a,i} \geq j+2) = o(\Delta t)$$

$$\mathbb{P}(I_{b,i+1} = k | I_{b,i} \geq k+2) = o(\Delta t)$$

Ignoring little-o terms, we can write the probability of having inventory levels (j, k) at time

$i+1$, written $P_{i+1,j,k}$, as a function of $P_{i+1,j,k}$, $P_{i+1,j+1,k}$, $P_{i+1,j,k+1}$, and $P_{i+1,j+1,k+1}$ as follows

$$P_{i+1,j,k} = \begin{cases} [1-\lambda_{a,i,j,k}\Delta t + o(\Delta t)] * [1-\lambda_{b,i,j,k}\Delta t + o(\Delta t)] * P_{i,j,k} & j = I_{a,0}, k = I_{b,0} \\ [1-\lambda_{a,i,j,k}\Delta t + o(\Delta t)] * [1-\lambda_{b,i,j,k}\Delta t + o(\Delta t)] * P_{i,j,k} & \\ \quad + [1-\lambda_{a,i,j,k+1}\Delta t + o(\Delta t)] * [\lambda_{b,i,j,k+1}\Delta t + o(\Delta t)] * P_{i,j,k+1} & j = I_{a,0}, 0 < k < I_{b,0} \\ [1-\lambda_{a,i,j,k}\Delta t + o(\Delta t)] * [1-\lambda_{b,i,j,k}\Delta t + o(\Delta t)] * P_{i,j,k} & \\ \quad + [\lambda_{a,i,j+1,k}\Delta t + o(\Delta t)] * [1-\lambda_{b,i,j+1,k}\Delta t + o(\Delta t)] * P_{i,j+1,k} & 0 \le j < I_{a,0}, k = I_{b,0} \\ [1-\lambda_{a,i,j,k}\Delta t + o(\Delta t)] * [1-\lambda_{b,i,j,k}\Delta t + o(\Delta t)] * P_{i,j,k} & \\ \quad + [1-\lambda_{a,i,j,k+1}\Delta t + o(\Delta t)] * [\lambda_{b,i,j,k+1}\Delta t + o(\Delta t)] * P_{i,j,k+1} & \\ \quad + [\lambda_{a,i,j+1,k}\Delta t + o(\Delta t)] * [1-\lambda_{b,i,j+1,k}\Delta t + o(\Delta t)] * P_{i,j+1,k} & \\ \quad + [\lambda_{a,i,j+1,k+1}\Delta t + o(\Delta t)] * [\lambda_{b,i,j+1,k+1}\Delta t + o(\Delta t)] * P_{i,j+1,k+1} & 0 \le j < I_{a,0}, 0 \le k < I_{b,0} \end{cases} \tag{4.19}$$

Using the initial probability state

$$\mathbf{P}(\mathbf{0}) = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix}, \tag{4.20}$$

we can forward iterate to find all values of $P_{i,j,k}$ including $P_{n,j,k}$. We can now find $\mathbb{E}[I_T]$, and consequently find the break even option price.

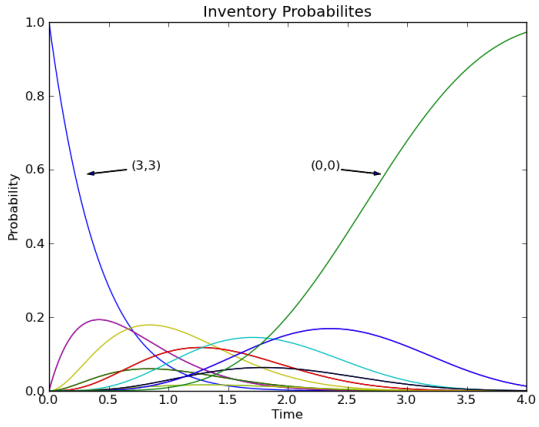

Figure 4.9: Inventory probabilities of duopolistic market with substitutes. Assume $I_{a,0} = I_{b,0} = 3$, $\lambda_1(p_1, p_2) = 10 - 5p_1 + 3p_2$, $\lambda_2(p_1, p_2) = 10 + 3p_1 - 5p_2$, $C_{a,0} = C_{b,0} = 4$, $C_{a,T} = C_{a,T} = 1$, $K_a = K_b = 2$ and $T = 4$.

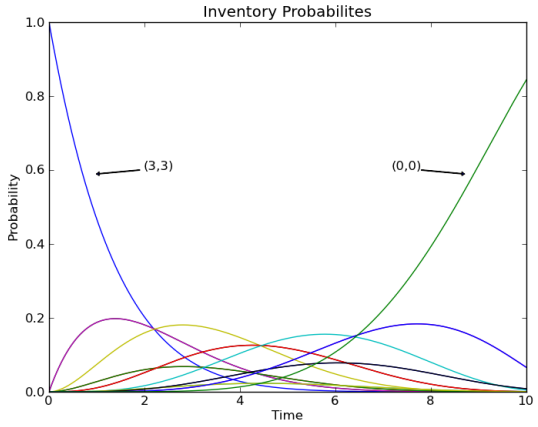

Figure 4.10: Inventory probabilities of a duopolistic market with complementary goods. Assume $I_{a,0} = I_{b,0} = 3$, $\lambda_1(p_1, p_2) = 10 - 2p_1 - p_2$, $\lambda_2(p_1, p_2) = 10 - p_1 - 2p_2$, $C_{a,0} = C_{b,0} = 3$, $C_{a,T} = C_{a,T} = 2$, $K_a = K_b = 2$, and $T = 10$.

Figures 4.9 and 4.10, we see the probabilities of being in any given state. In both cases, there's an initial inventory level of $(3, 3)$ but as time progresses, the probability of having inventory levels of $(0, 0)$ increases. Every other inventory state has some increase at the beginning of the time interval and then drops off as time proceeds. As we'd expect, the more time we have the inventory on sale, the higher the probability we've sold it all.

Given the final inventory probabilities we can now calculate $\mathbb{E}[I_T]$ and p (the price of the inventory option) for both the substitutes and the complements. We obtain $\mathbb{E}[I_T] = 0.0146$ and $p = (K - C_T)\mathbb{E}[I_T] = 0.0146$ for the substitutes and $\mathbb{E}[I_T] = 0.0911$ and $p = (K - C_T)\mathbb{E}[I_T] = 0.0911$ for the complements.

Chapter 5. Duopolistic Newsvendor Problem with Collusion

We have found the optimal inventory levels, optimal sale price strategies, the expected profit, inventory probability progression, and the cross price elasticity of one firm in a two firm market. Now we look at the case when the companies collude. When two companies collude, they team up and try to optimize the total profit of both firms. For example, if there are only two newsvendors, then they may get together in order to maximize their joint expected

profit. We could consider the profit function as a total profit from both goods instead of just examining the profit of an individual company, as we did before, and then split the profits between the two firms.

Since the total profit function is just the revenues of the whole market minus the total cost, we essential are looking at a monopolistic market. Because the same results will hold for a monopolistic market, we will just consider the profit function from the standpoint of a monopoly. Another application of this would be if there was one company that sold two different goods or one newsvendor that set up shops in two different places.

Generally when two companies collude, one already has more market power and has leverage on the other company. Due to this leverage, the profits are not always split $50:50$. We set up the total profit function as a function of both goods and note that the negotiations between the two firms will determine the percentage of profits to each firm.

Again we partition the time interval, $[0,T]$, into n equal intervals so that $T = n\Delta t$ and write the profit and expect profit.

$$\begin{aligned}
\Pi = \quad & \sum_{k=0}^{n-1} \left[u_{a,k,I_{a,k},I_{b,k}} S(I_{a,k}, D_{a,k}) + u_{b,k,I_{a,k},I_{b,k}} S(I_{b,k}, D_{b,k}) \right] \\
& + C_{a,T} I_{a,T} + C_{b,T} I_{b,T} - C_{a,0} I_{a,0} - C_{b,0} I_{b,0} \\
\mathbb{E}[\Pi] = \quad & \sum_{k=0}^{n-1} \left[u_{a,k,I_{a,k},I_{b,k}} \mathbb{E}[S(I_{a,k}, D_{a,k})] + u_{b,k,I_{a,k},I_{b,k}} \mathbb{E}[S(I_{b,k}, D_{b,k})] \right] \\
& + C_{a,T} \mathbb{E}[I_{a,T}] + C_{b,T} \mathbb{E}[I_{b,T}] - C_{a,0} I_{a,0} - C_{b,0} I_{b,0}
\end{aligned}$$

We also see that total revenue and remaining revenue functions are given as follows

$$\begin{aligned} R = & \sum_{k=0}^{n-1} \left[u_{a,k,I_{a,k},I_{b,k}} S(I_{a,k}, D_{a,k}) + u_{b,k,I_{a,k},I_{b,k}} S(I_{b,k}, D_{b,k}) \right] \\ & + C_{a,T} I_{a,T} + C_{b,T} I_{b,T} \\ R(i\Delta t) = & \sum_{k=i}^{n-1} \left[u_{a,k,I_{a,k},I_{b,k}} \mathbb{E}[S(I_{a,k}, D_{a,k})] + u_{b,k,I_{a,k},I_{b,k}} \mathbb{E}[S(I_{b,k}, D_{b,k})] \right] \\ & + C_{a,T} \mathbb{E}[I_{a,T}] + C_{b,T} \mathbb{E}[I_{b,T}] \end{aligned}$$

Our demand functions for the individual goods are the same as before and given in Equation 4.2. The sales equation for S_a and S_b are the same as in Equation 4.1 as well as the state equation for $I_{a,i+1}$ and $I_{b,i+1}$. Following the same reasoning as above, our expected remaining revenue is still given in Equation 4.3 by

$$\begin{aligned} E_{a,i,j,k} &= \mathbb{E}(\mathbb{E}[R_{a,i,j,k} | D_{a,i}, D_{b,i}]) \\ &= \sum_{c,d \geq 0} \mathbb{E}[R_{a,i,j,k} | D_{a,i} = c, D_{b,i} = d] \mathbb{P}(D_{a,i} = c, D_{b,i} = d) \\ &= \mathbb{E}[R_{a,i,j,k} | D_{a,i}, D_{b,i} = 0] \mathbb{P}(D_{a,i}, D_{b,i} = 0) \\ &\quad + \mathbb{E}[R_{a,i,j,k} | D_{a,i} = 0, D_{b,i} = 1] \mathbb{P}(D_{a,i} = 0, D_{b,i} = 1) \\ &\quad + \mathbb{E}[R_{a,i,j,k} | D_{a,i} = 1, D_{b,i} = 0] \mathbb{P}(D_{a,i} = 1, D_{b,i} = 0) \\ &\quad + \mathbb{E}[R_{a,i,j,k} | D_{a,i}, D_{b,i} = 1] \mathbb{P}(D_{a,i}, D_{b,i} = 1) \\ &\quad + o(\Delta t). \end{aligned} \tag{5.1}$$

But we can now show that the Equations 4.4-4.7 change to

$$\mathbb{E}[R_{i,j,k} | D_{a,i} = 0, D_{b,i} = 0] = E_{a,i+1,j,k} \tag{5.2}$$

$$\mathbb{E}[R_{i,j,k} | D_{a,i} = 0, D_{a,i} = 1] = u_{b,i,j,k} + E_{a,i+1,j,k-1} \tag{5.3}$$

$$\mathbb{E}[R_{i,j,k} | D_{a,i} = 1, D_{a,i} = 0] = u_{a,i,j,k} + E_{a,i+1,j,k-1} \tag{5.4}$$

$$\mathbb{E}[R_{i,j,k} | D_{a,i} = 1, D_{a,i} = 1] = u_{a,i,j,k} + u_{b,i,j,k} + E_{a,i+1,j-1,k-1} \tag{5.5}$$

Our $P(D_{a,i}, D_{b,i} = 0), P(D_{a,i} = 0, D_{b,i} = 1), P(D_{a,i} = 1, D_{b,i} = 0)$, and $P(D_{a,i}, D_{b,i} = 1)$ are all given in Equations 4.8-4.11. Substituting Equations 4.8-4.11 and 5.2-5.5 into Equation 5.1 gives us

$$\begin{aligned} E_{i,j,k} &= [1 - \lambda_{a,i,j,k}\Delta t] * [1 - \lambda_{b,i,j,k}\Delta t] (E_{i+1,j,k}) \\ &\quad + [1 - \lambda_{a,i,j,k}\Delta t] * \lambda_{b,i,j,k}\Delta t * (u_{b,i,j,k} + E_{i+1,j,k-1}) \\ &\quad + \lambda_{a,i,j,k}\Delta t * [1 - \lambda_{b,i,j,k}\Delta t] (u_{a,i,j,k} + E_{i+1,j-1,k}) \\ &\quad + \lambda_{a,i,j,k}\Delta t * \lambda_{b,i,j,k}\Delta t (u_{a,i,j,k} + u_{b,i,j,k} + E_{i+1,j-1,k-1}) \\ &\quad + o(\Delta t) \end{aligned}$$

We know that $R_{n,j,k} = jC_{a,T} + kC_{b,T}$ where n is the number of discrete time intervals. Find $E_{i,j,k}$ for all i, j, k by using a dynamical program to iterate backwards from this initial condition. If j or k is zero, then the corresponding Poisson rate is zero. We can write $\lambda_{a,i,0,k} = 0$ and $\lambda_{b,i,j,0} = 0$ and get equations for expected remaining revenue when one good has zero units of inventory left.

$$\begin{aligned} E_{i,j,0} &= [1 - \lambda_{a,i,j,0}\Delta t] * (E_{i+1,j,0}) + \lambda_{a,i,j,0}\Delta t * (u_{a,i,j,0} + E_{i+1,j-1,0}) + o(\Delta t) \\ E_{i,0,k} &= [1 - \lambda_{b,i,0,k}\Delta t] * (E_{i+1,0,k}) + \lambda_{b,i,0,k}\Delta t * (u_{b,i,0,k} + E_{i+1,0,k-1}) + o(\Delta t) \end{aligned}$$

5.1 Pricing Strategy

To find the optimal price for inventory levels j, k at time i, we take the derivative of the previous equations with respect to $u_{a,i,j,k}$ and $u_{b,i,j,k}$, set them equal to zero, take the limit as Δt approaches zero and solve for $u^*_{a,i,j,k}$ and $u^*_{b,i,j,k}$. Through symmetry we can infer that $u^*_{a,i,j,k}$ and $u^*_{b,i,j,k}$ will be similar.

When $k = 0$, we assume the price of the other good is equal to its last price which occurs when there was still a unit of inventory. We do this because there's no product to sell, so

it doesn't make sense to vary it to optimize the problem. We could send the price higher because having no inventory is the same as sending the price to infinity (it's impossible to buy in both cases). For computational simplicity, we set the price of the zero inventory level good to its 'previous' price before selling the last good. So we keep the optimal price of good a as a function of the price of good b since we're not varying good b's price. We find

$$0 = \frac{\partial E_{i,j,0}}{\partial u_{a,i,j,0}} = -\frac{\partial \lambda_{a,i,j,0}}{\partial u_{a,i,j,0}} \Delta t * (E_{i+1,j,0}) + \frac{\partial \lambda_{a,i,j,0}}{\partial u_{a,i,j,0}} \Delta t * (u_{a,i,j,0} + E_{i+1,j-1,0}) + \lambda_{a,i,j,0} \Delta t$$

Dividing by Δt, we get

$$0 = -\frac{\partial \lambda_{a,i,j,0}}{\partial u_{a,i,j,0}} (E_{i+1,j,0}) + \frac{\partial \lambda_{a,i,j,0}}{\partial u_{a,i,j,0}} (u_{a,i,j,0} + E_{i+1,j-1,0}) + \lambda_{a,i,j,0} \tag{5.6}$$

Similarly, we get

$$0 = -\frac{\partial \lambda_{b,i,0,k}}{\partial u_{b,i,0,k}} (E_{i+1,0,k}) + \frac{\partial \lambda_{b,i,0,k}}{\partial u_{b,i,0,k}} (u_{b,i,0,k} + E_{i+1,0,k-1}) + \lambda_{b,i,0,k} \tag{5.7}$$

when $j = 0$.

When $j, k \neq 0$, we take the derivative of $\mathbb{E}_{i,j,k}$ with respect to $u_{a,i,j,k}$ to get the following:

$$\begin{aligned}
0 = \frac{\partial E_{i,j,k}}{\partial u_{a,i,j,k}} &= -\frac{\partial \lambda_{a,i,j,k}}{\partial u_{a,i,j,k}} \Delta t \left[1 - \lambda_{b,i,j,k}\Delta t\right] \left(E_{i+1,j,k}\right) \\
&\quad - \left[1 - \lambda_{a,i,j,k}\Delta t\right] \frac{\partial \lambda_{b,i,j,k}}{\partial u_{a,i,j,k}} \Delta t \left(E_{i+1,j,k}\right) \\
&\quad - \frac{\partial \lambda_{a,i,j,k}}{\partial u_{a,i,j,k}} \Delta t * \lambda_{b,i,j,k}\Delta t * \left(u^*_{b,i,j,k} + E_{i+1,j,k-1}\right) \\
&\quad + \left[1 - \lambda_{a,i,j,k}\Delta t\right] * \frac{\partial \lambda_{b,i,j,k}}{\partial u_{a,i,j,k}} \Delta t * \left(u^*_{b,i,j,k} + E_{i+1,j,k-1}\right) \\
&\quad + \frac{\partial \lambda_{a,i,j,k}}{\partial u_{a,i,j,k}} \Delta t * \left[1 - \lambda_{b,i,j,k}\Delta t\right] \left(u^*_{a,i,j,k} + E_{i+1,j-1,k}\right) \\
&\quad - \lambda_{a,i,j,k}\Delta t * \frac{\partial \lambda_{b,i,j,k}}{\partial u_{a,i,j,k}} \Delta t \left(u^*_{a,i,j,k} + E_{i+1,j-1,k}\right) \\
&\quad + \lambda_{a,i,j,k}\Delta t * \left[1 - \lambda_{b,i,j,k}\Delta t\right] \\
&\quad + \frac{\partial \lambda_{a,i,j,k}}{\partial u_{a,i,j,k}} \Delta t * \lambda_{b,i,j,k}\Delta t \left(u^*_{a,i,j,k} + u^*_{b,i,j,k} + E_{i+1,j-1,k-1}\right) \\
&\quad + \lambda_{a,i,j,k}\Delta t * \frac{\partial \lambda_{b,i,j,k}}{\partial u_{a,i,j,k}} \Delta t \left(u^*_{a,i,j,k} + u^*_{b,i,j,k} + E_{i+1,j-1,k-1}\right) \\
&\quad + \lambda_{a,i,j,k}\Delta t * \lambda_{b,i,j,k}\Delta t
\end{aligned}$$

Divide by Δt, take the limit as $\Delta t \to 0$, and simplify to get

$$\begin{aligned}
0 &= -\frac{\partial \lambda_{a,i,j,k}}{\partial u_{a,i,j,k}} \left(E_{i+1,j,k}\right) - \frac{\partial \lambda_{b,i,j,k}}{\partial u_{a,i,j,k}} \left(E_{i+1,j,k}\right) \\
&\quad + \frac{\partial \lambda_{b,i,j,k}}{\partial u_{a,i,j,k}} \left(u^*_{b,i,j,k} + E_{i+1,j,k-1}\right) + \frac{\partial \lambda_{a,i,j,k}}{\partial u_{a,i,j,k}} \left(u^*_{a,i,j,k} + E_{i+1,j-1,k}\right) + \lambda_{a,i,j,k}. \qquad (5.8)
\end{aligned}$$

A similar equation holds for $0 = \frac{\partial E_{i,j,k}}{\partial u_{b,i,j,k}}$.

5.1.1 Linear Demand. Let the Poisson parameters be linear functions with respect to the prices.

$$\begin{bmatrix} \lambda_1(t) \\ \lambda_2(t) \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} - \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix}$$

where $b_{11}, b_{22} > 0$. We can now find the optimal pricing strategy for the linear demand case by substituting in for $\lambda_{a,i,j,0}$ and $\frac{\partial \lambda_{a,i,j,k}}{\partial u_{a,i,j,k}} = -b_{11}$ into Equation 5.6.

$$\begin{aligned} 0 &= b_{11}\left(E_{a,i+1,j,0}\right) - b_{11}\left(u^*_{a,i,j,0} + E_{a,i+1,j-1,0}\right) + a_1 - b_{11}u^*_{a,i,j,0} - b_{12}u_{b,i,j,0} \\ u^*_{a,i,j,0} &= \frac{1}{2b_{11}}\left(a_1 + b_{11}(E_{a,i+1,j,0} - E_{a,i+1,j-1,0}) - b_{12}u_{b,i,j,0}\right) \end{aligned}$$

Similarly, from Equation 5.7, we know

$$u^*_{b,i,0,k} = \frac{1}{2b_{22}}\left(a_2 + b_{22}(E_{b,i+1,0,k} - E_{b,i+1,0,k-1}) - b_{21}u_{a,i,0,k}\right).$$

Now substitute $\lambda_{a,i,j,0}$ and $\frac{\partial \lambda'_{a,i,j,k}}{\partial u_{a,i,j,k}}$ into Equation 5.8 to get

$$\begin{aligned} 0 &= b_{11}\left(E_{a,i+1,j,k}\right) + b_{22}\left(E_{a,i+1,j,k}\right) - b_{22}\left(E_{a,i+1,j,k-1}\right) - b_{11}\left(u^*_{a,i,j,k} + E_{a,i+1,j-1,k}\right) \\ &\quad + (a_1 - b_{11}u^*_{a,i,j,k} - b_{12}u_{b,i,j,k}) \\ u^*_{a,i,j,k} &= \frac{1}{2b_{11}}\left(a_1 + b_{11}E_{a,i+1,j,k} + b_{21}E_{a,i+1,j,k} - b_{21}E_{a,i+1,j,k-1} - b_{11}E_{a,i+1,j-1,k} - b_{12}u_{b,i,j,k}\right) \end{aligned}$$

By symmetry we can find a similar optimal value for $u^*_{b,i,j,k}$. Set up the system of equations and solve for $u^*_{a,i,j,k}$ and $u^*_{b,i,j,k}$ to get

$$\begin{bmatrix} u^*_{a,i,j,k} \\ u^*_{b,i,j,k} \end{bmatrix} = \begin{bmatrix} 2b_{11}b_{22} & b_{22}(b_{12}+b_{21}) \\ b_{11}(b_{12}+b_{21}) & 2b_{11}b_{22} \end{bmatrix}^{-1} \begin{bmatrix} b_{22}\left(a_1 + (b_{11}+b_{21})E_{a,i+1,j,k} - b_{21}E_{a,i+1,j,k-1} - b_{11}E_{a,i+1,j-1,k}\right) \\ b_{11}\left(a_2 + (b_{22}+b_{12})E_{b,i+1,j,k} - b_{12}E_{b,i+1,j-1,k} - b_{22}E_{b,i+1,j,k-1}\right) \end{bmatrix}$$

whenever $j, k \neq 0$.

The Figures 5.1 and 5.3 are the graphs of the optimal pricing strategies of a two good market with linear demand functions. The first graph is when the goods are substitutes and the second graph shows complementary goods. Figures 5.2 and 5.4. We see the same patterns as we did in the duopolistic market. That is, the sales price and the expected remaining revenue decrease as the time period nears the end. The optimal price strategy for the complementary goods doesn't dip like it did in the duopolistic market but it would with

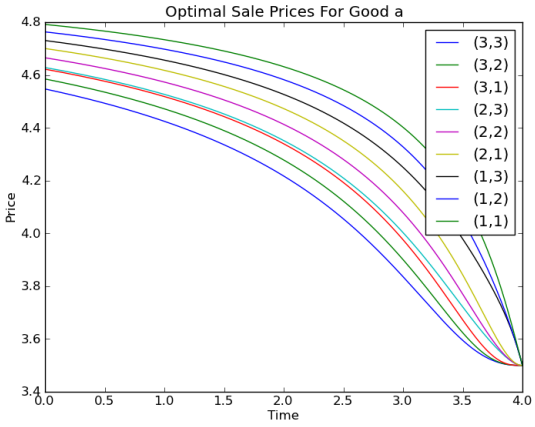

Figure 5.1: Optimal sale prices in a monopolistic market with substitutes. Assume $I_{a,0} = I_{b,0} = 3$, $\lambda_1(p_1, p_2) = 10 - 5p_1 + 3p_2$, $\lambda_2(p_1, p_2) = 10 + 3p_1 - 5p_2$, $C_{a,0} = C_{b,0} = 4$, $C_{a,T} = C_{a,T} = 2$, and $T = 4$.

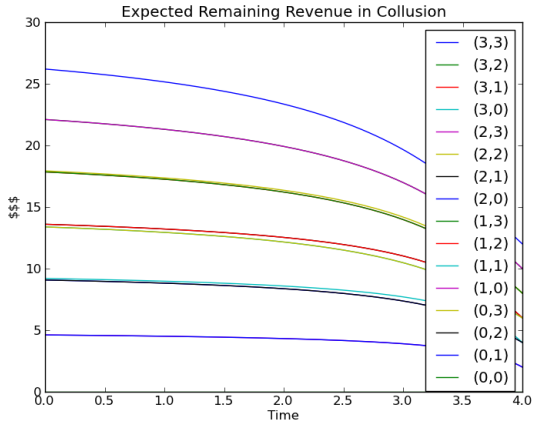

Figure 5.2: Expected remaining revenue in a monopolistic market with substitutes. Assume $I_{a,0} = I_{b,0} = 3$, $\lambda_1(p_1, p_2) = 10 - 5p_1 + 3p_2$, $\lambda_2(p_1, p_2) = 10 + 3p_1 - 5p_2$, $C_{a,0} = C_{b,0} = 4$, $C_{a,T} = C_{a,T} = 2$, and $T = 4$.

higher levels of initial inventory.

Figures 5.5 and 5.6 are the graphs of the cross price elasticity of the two different markets. We observe similar behavior as in the duopolistic market; however, there are small differences in the duopolistic and monopolistic cases. Because these differences in behavior are less intuitive and hard to explain, we do not try to explain them but only point them out.

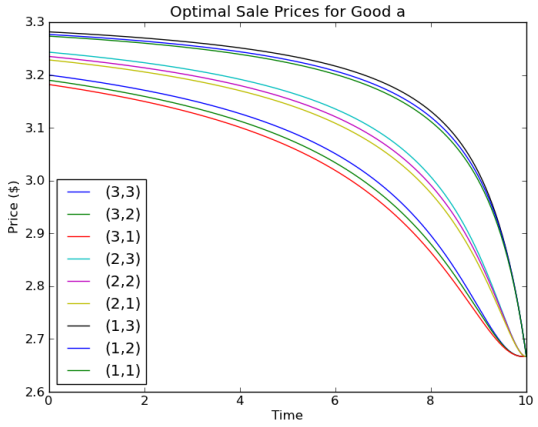

Figure 5.3: Optimal sale prices in a monopolistic market for complementary goods. Assume $I_{a,0} = I_{b,0} = 4,\ \lambda_1(p_1, p_2) = 10 - 2p_1 - p_2,\ \lambda_2(p_1, p_2) = 10 - p_1 - 2p_2,\ C_{a,0} = C_{b,0} = 3,\ C_{a,T} = C_{a,T} = 2$, and $T = 10$.

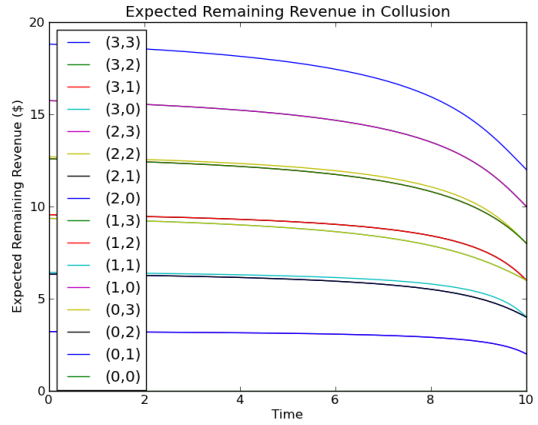

Figure 5.4: Expected remaining revenue in a monopolistic market for complementary goods. Assume $I_{a,0} = I_{b,0} = 4,\ \lambda_1(p_1, p_2) = 10 - 2p_1 - p_2,\ \lambda_2(p_1, p_2) = 10 - p_1 - 2p_2,\ C_{a,0} = C_{b,0} = 3,\ C_{a,T} = C_{a,T} = 2$, and $T = 10$.

In the substitutes case, the cross price elasticities seem to be more dependent on the other good than they did in the duopolistic case. We also observe for the complementary goods, each inventory level's magnitude for the monopolistic market seems to be greater than that of the duopolistic market.

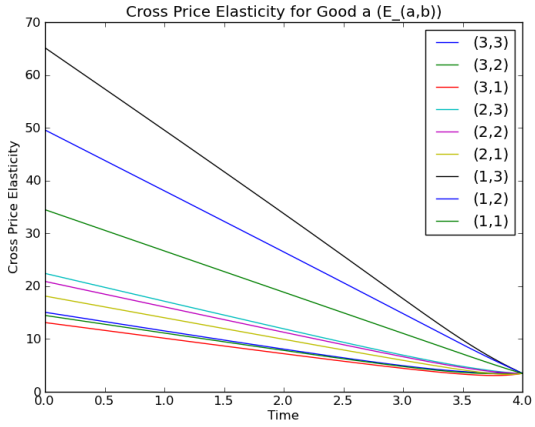

Figure 5.5: Cross price elasticity in a monopolistic market with substitutes. Assume $I_{a,0} = I_{b,0} = 3$, $\lambda_1(p_1, p_2) = 10 - 5p_1 + 3p_2$, $\lambda_2(p_1, p_2) = 10 + 3p_1 - 5p_2$, $C_{a,0} = C_{b,0} = 4$, $C_{a,T} = C_{a,T} = 2$, and $T = 4$.

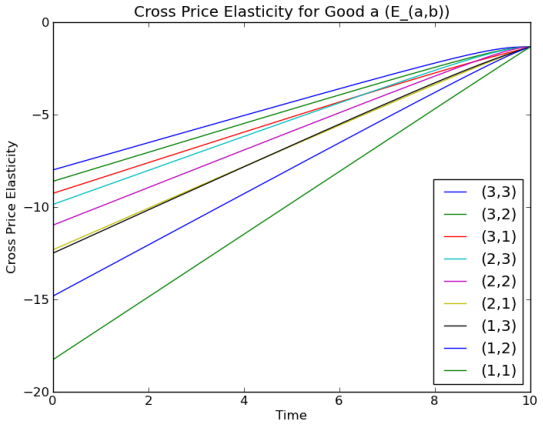

Figure 5.6: Cross price elasticity in a monopolistic market for complementary goods. Assume $I_{a,0} = I_{b,0} = 4$, $\lambda_1(p_1, p_2) = 10 - 2p_1 - p_2$, $\lambda_2(p_1, p_2) = 10 - p_1 - 2p_2$, $C_{a,0} = C_{b,0} = 3$, $C_{a,T} = C_{a,T} = 2$, and $T = 10$.

5.2 Optimal Inventory

Again we desire to find the optimal inventory level that maximizes profits for the firm. Now that we are considering a monopolistic framework, we look at the optimal inventory level of both goods instead of our previous approach of finding a Nash Equilibrium.

We solve for the optimal inventory level with the same approach as before. Recall that

Equation 4.16 gave us the total expected profit as the expected remaining revenue minus the total costs

$$\mathbb{E}[\Pi_a] = \mathbb{E}[R_a] - C_{a,0}I_{a,0}.$$

We use the expected revenue at the initial time, $R_{0,j,k}$, given a fixed inventory level, (j,k), from the previous section and we subtract the total cost, $C_{a,0}I_{a,0} + C_{b,0}I_{b,0}$, to find the expected profit. As in the duopolistic case, we numerically find the optimal inventory levels by plotting the expected profits over the initial inventory levels. Figures 5.7 and 5.8 are graphs of the expected profits over the initial inventory levels for product a.

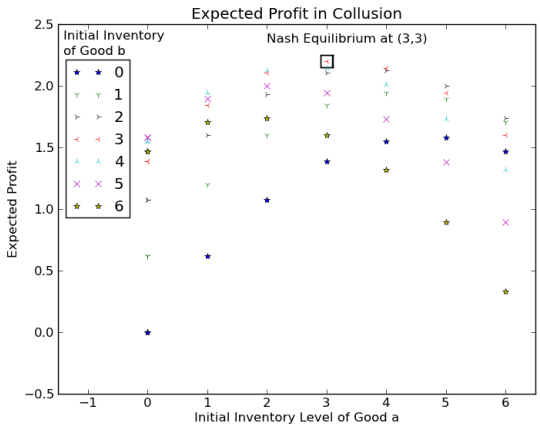

Figure 5.7: Optimal initial inventory in a monopolistic market for substitutes. Assume $I_{a,0} = I_{b,0} = 3$, $\lambda_1(p_1,p_2) = 10 - 5p_1 + 3p_2$, $\lambda_2(p_1,p_2) = 10 + 3p_1 - 5p_2$, $C_{a,0} = C_{b,0} = 4$, $C_{a,T} = C_{a,T} = 2$, and $T = 4$.

We see that the optimal initial inventory for both the substitutes and the complements are both $(3,3)$ in both the duopolistic market and the monopolistic market. This is an interesting phenomenon because generally we'd expected the number of units sold to decrease in the monopolistic case. But in both these cases, the monopolistic produces the same amount of initial inventory as the duopolistic cases.

Another thing we notice is that for the substitutes, the expected profit for the monopolistic market is much higher than the sum of the profits for both firms in the duopolistic market. In fact we see a combined total of about 1.6 units of profit in the duopolistic case

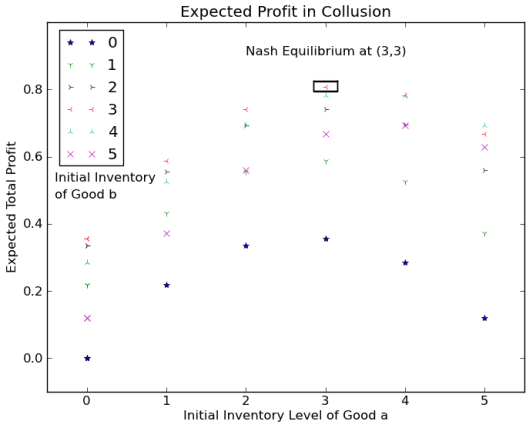

Figure 5.8: Optimal initial inventory in a monopolistic market for complementary goods. Assume $I_{a,0} = I_{b,0} = 4$, $\lambda_1(p_1, p_2) = 10 - 2p_1 - p_2$, $\lambda_2(p_1, p_2) = 10 - p_1 - 2p_2$, $C_{a,0} = C_{b,0} = 3$, $C_{a,T} = C_{a,T} = 2$, and $T = 10$.

and nearly 2.25 units of profit in the monopolistic case. We expected to see this because traditional economics teaches us that monopolies always have significantly higher profits than the combined profits of the oligopolistic firms.

The interesting thing is that this is not the case for the complementary goods. Although the monopolistic firms gains slightly higher profits than the combined duopolistic firms' profits, it is not nearly as significant as the gain in profits in the substitutes case. The combined total profits in the duopolistic market is roughly 0.75 units; whereas, the profits in the monopolistic market are around 0.80.

5.3 Inventory Probabilities

The only difference in calculating the inventory probabilities in the duopolistic case and the monopolistic case is the difference in $\lambda_{a,i,j,k}$ and $\lambda_{b,i,j,k}$. However, these values are all calculated while finding the optimal sale price strategies and the expected remaining revenues. So once again we can calculate the inventory probabilities using the initial probability state given in Equation 4.20 and the probability update formula in Equation 4.19.

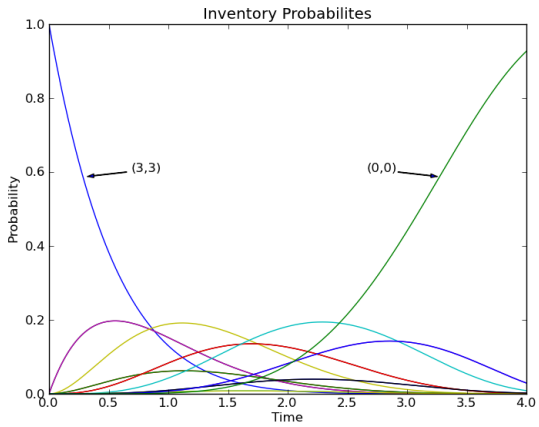

Figure 5.9: Inventory probabilities in a monopolistic market with substitutes. Assume $I_{a,0} = I_{b,0} = 3$, $\lambda_1(p_1, p_2) = 10 - 5p_1 + 3p_2$, $\lambda_2(p_1, p_2) = 10 + 3p_1 - 5p_2$, $C_{a,0} = C_{b,0} = 4$, $C_{a,T} = C_{a,T} = 1$, $K_a = K_b = 2$, and $T = 4$.

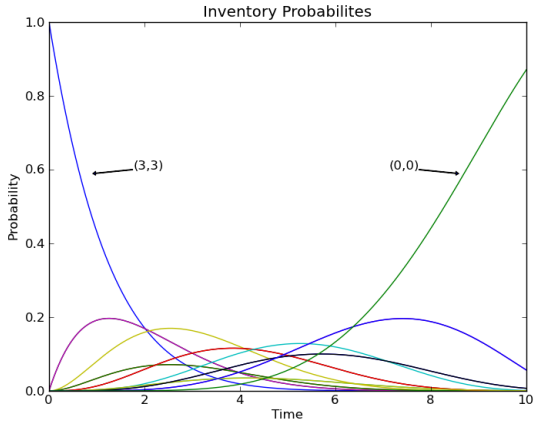

Figure 5.10: Inventory probabilities in a monopolistic market with complementary goods. Assume $I_{a,0} = I_{b,0} = 3$, $\lambda_1(p_1, p_2) = 10 - 2p_1 - p_2$, $\lambda_2(p_1, p_2) = 10 - p_1 - 2p_2$, $C_{a,0} = C_{b,0} = 3$, $C_{a,T} = C_{a,T} = 1$, $K_a = K_b = 2$, and $T = 10$.

We once again find progress of the inventory probabilities and graph them in Figures 5.9 and 5.10. We use the results to find the expected remaining inventory for each good and the corresponding option price as given in Equations 4.17 and 4.18. We find that in our substitute case, we have $\mathbb{E}[I_{a,T}] = \mathbb{E}[I_{b,T}] = 0.04534$ and the option price is $p_a = p_b = 0.04534$; whereas in our complementary goods case, we have $\mathbb{E}[I_{a,T}] = \mathbb{E}[I_{b,T}] = 0.07303$ and the option price is $p_a = p_b = 0.07303$.

Chapter 6. Differentiated Goods

Until now, we have only considered homogeneous goods. That is to say that the demand function is identical for both firms. We would, however, like to know what would happen in the case of heterogeneous goods or differentiated goods. In the context of the newsvendor problem, this could arise if there are two different qualities of newspapers. For example, we could consider a duopolistic market comprised of the sales of a renowned paper such as the New York Times in comparison to a community paper. Clearly we'd expect the better known paper to have a higher sales price but we don't know about the relationship of the demand between them. For example, the New York Times might sell more in most cities than the local paper but maybe in some communities, the local paper is in higher demand than the New York Times.

We might also consider a monopolistic market such as the sales of two different automobiles made by the same company. For example we could consider the sales of a Toyota Corolla versus a Lexus IS (both owned by Toyota). Clearly a Lexus is a more expensive car but Toyota Corolla's sell much more quantity than the Lexus IS's do. A business may want to know how to price the two goods in order to capture the maximum possible profits.

We will examine the monopolistic and duopolistic cases of differentiated goods both substitutes and complements, noting that the monopolistic cases encompasses collusion in a duopolistic market. We will first start by separating into the substitutes and complements cases. We will then subdivide each of those sections into the monopolistic and duopolistic cases.

6.1 Substitutes

We first examine what happens when the goods are substitutes. Say the demand of the goods are random variables modeled with Poisson processes, the goods are substitutes, and assume the demand for the Poisson rates is linear and given by the system of equations below

$$\begin{bmatrix} \lambda_1(t) \\ \lambda_2(t) \end{bmatrix} = \begin{bmatrix} 15 \\ 20 \end{bmatrix} - \begin{bmatrix} 4 & -1 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix}.$$

Let the initial and salvage costs of the goods be $C_{a,0} = 3.5$, $C_{b,0} = 1$, $C_{a,T} = 2$, and $C_{b,T} = 0.5$. Consider first a duopolistic model and then a monopolistic model.

6.1.1 Duopolistic. Consider a duopolistic market with two different firms exclusively selling their respective goods. We first plot the expected profit in order to find the Nash equilibrium for initial inventory levels.

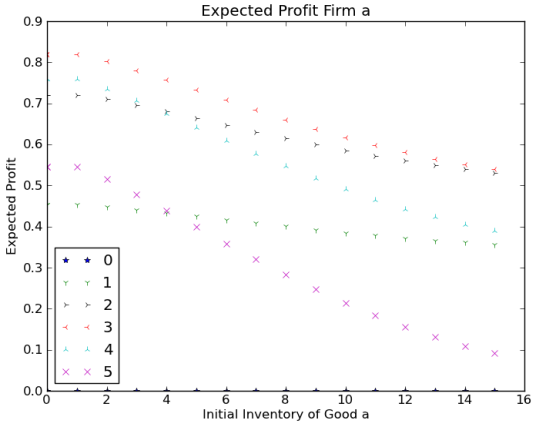

Figure 6.1: Expected profit of firm one in a duopolistic market with differentiated, complementary goods. Let $\lambda_1(p_1, p_2) = 15 - 4p_1 + 1p_2$, $\lambda_2(p_1, p_2) = 10 + 2p_1 - 7p_2$, $C_{a,0} = 3.5$, $C_{b,0} = 1$, $C_{a,T} = 2$, and $C_{b,T} = 0.5$, and $T = 3$.

In Figure 6.1, it shows that whenever firm b starts with less than 15 units of inventory, firm a's optimal initial inventory is three units. Figure 6.2 suggests that whenever firm a has three units of inventory, firm b maximizes optimal profit at initial inventory level of 9. So the Nash Equilibrium is found at $(I_{a,0}, I_{b,0}) = (3, 9)$. We now use previous techniques to find

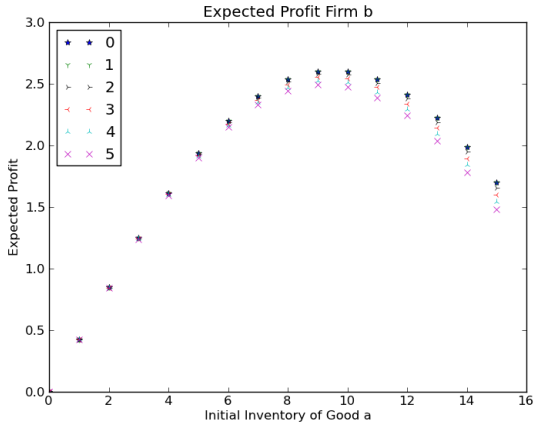

Figure 6.2: Expected profit of firm two in a duopolistic market with differentiated, complementary goods. Let $\lambda_1(p_1, p_2) = 15 - 4p_1 + 1p_2$, $\lambda_2(p_1, p_2) = 10 + 2p_1 - 7p_2$, $C_{a,0} = 3.5$, $C_{b,0} = 1$, $C_{a,T} = 2$, and $C_{b,T} = 0.5$, and $T = 3$.

the optimal sales price strategies, the cross price elasticities, and the expected remaining revenues for both firms, a and b. Since the progression of the inventory probabilities looks similar to our other examples, we omit it from now on.

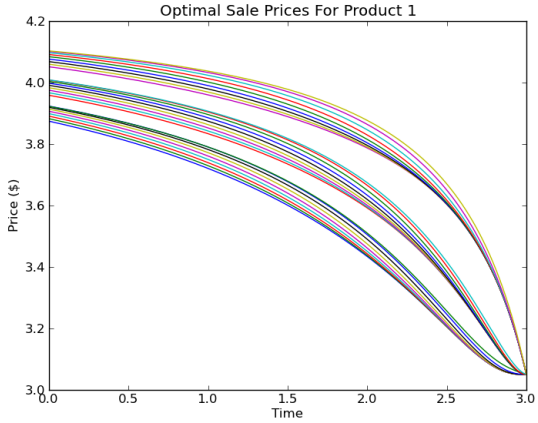

Figure 6.3: Optimal sales price strategy of firm one in a duopolistic market with differentiated, substitutes. Let $\lambda_1(p_1, p_2) = 15 - 4p_1 + 1p_2$, $\lambda_2(p_1, p_2) = 10 + 2p_1 - 7p_2$, $C_{a,0} = 3.5$, $C_{b,0} = 1$, $C_{a,T} = 2$, and $C_{b,T} = 0.5$, and $T = 3$.

Figures 6.3-6.8 are similar to what we'd expect and have seen in previous examples. The only thing we wish to point out is the effect of the market leverage between the two goods. We have set the market up so that good a will have more leverage than good b. This is seen in the price coefficients, b_{12} and b_{21}, in the demand function. The effects of the difference

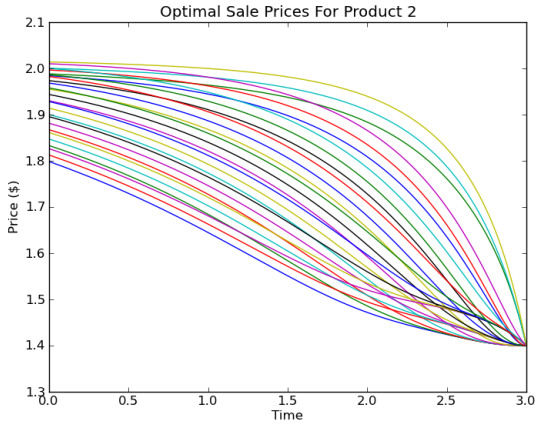

Figure 6.4: Optimal sales price strategy of firm two in a duopolistic market with differentiated, substitutes. Let $\lambda_1(p_1, p_2) = 15 - 4p_1 + 1p_2$, $\lambda_2(p_1, p_2) = 10 + 2p_1 - 7p_2$, $C_{a,0} = 3.5$, $C_{b,0} = 1$, $C_{a,T} = 2$, and $C_{b,T} = 0.5$, and $T = 3$.

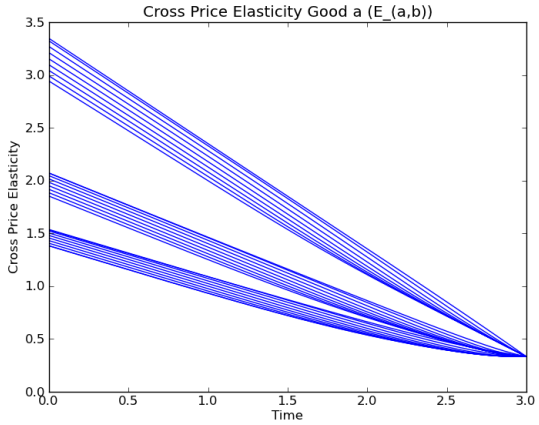

Figure 6.5: Cross price elasticities of good a in a duopolistic market with differentiated, substitutes. Let $\lambda_1(p_1, p_2) = 15 - 4p_1 + 1p_2$, $\lambda_2(p_1, p_2) = 10 + 2p_1 - 7p_2$, $C_{a,0} = 3.5$, $C_{b,0} = 1$, $C_{a,T} = 2$, and $C_{b,T} = 0.5$, and $T = 3$.

in market leverage between the two goods is seen in the pricing strategies, the expected remaining revenue, and the cross price elasticity.

The sales price of the second good, firm b, depend much more on the prices of firm a. We can see this in Figures 6.3 and 6.4 because the sales price of good a is clumped together in groups of inventory levels of $(1, \cdot)$, $(2, \cdot)$, and $(3, \cdot)$. On the other hand, the sales prices of good b isn't clumped together in such a manner and this is due to the disparity in market leverage.

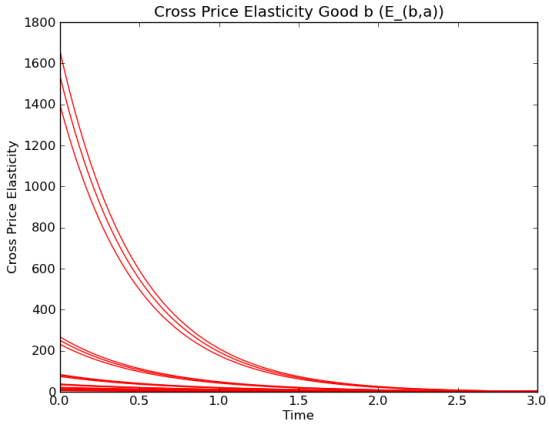

Figure 6.6: Cross price elasticitiies of good b in a duopolistic market with differentiated, substitutes. Let $\lambda_1(p_1, p_2) = 15 - 4p_1 + 1p_2$, $\lambda_2(p_1, p_2) = 10 + 2p_1 - 7p_2$, $C_{a,0} = 3.5$, $C_{b,0} = 1$, $C_{a,T} = 2$, and $C_{b,T} = 0.5$, and $T = 3$.

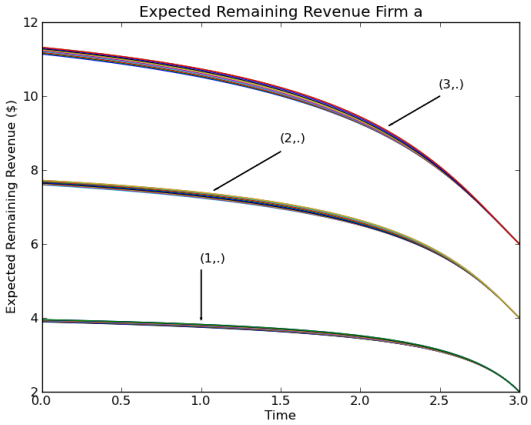

Figure 6.7: Expected remaining revenue of firm one in a duopolistic market with differentiated, complementary goods. Let $\lambda_1(p_1, p_2) = 15 - 4p_1 + 1p_2$, $\lambda_2(p_1, p_2) = 10 + 2p_1 - 7p_2$, $C_{a,0} = 3.5$, $C_{b,0} = 1$, $C_{a,T} = 2$, and $C_{b,T} = 0.5$, and $T = 3$.

We plot the two goods' cross price elasticities in Figures 6.5 and 6.5 because they are different now that we are examining differentiated goods. We see a negative correlation between the magnitude of cross price elasticity and market leverage. In other words, good b has higher cross price elasticity but has less market leverage.

The expected remaining revenue of firm b drops off faster than the expected remaining

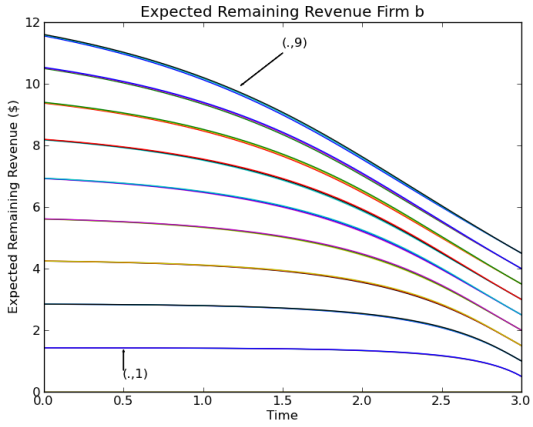

Figure 6.8: Expected remaining revenue of firm two in a duopolistic market with differentiated, complementary goods. Let $\lambda_1(p_1, p_2) = 15 - 4p_1 + 1p_2$, $\lambda_2(p_1, p_2) = 10 + 2p_1 - 7p_2$, $C_{a,0} = 3.5$, $C_{b,0} = 1$, $C_{a,T} = 2$, and $C_{b,T} = 0.5$, and $T = 3$.

revenue of firm a, as seen in Figures 6.7 and 6.8. This is in part due to the fact that there is more inventory for firm b so the firm must dump inventory sooner. However, we also attribute less market leverage to the decrease in expected remaining revenue of good b.

6.1.2 Monopolistic. Now consider the same demand structure in the monopolistic market. We first plot the expected profit in order to find the optimal initial inventory levels.

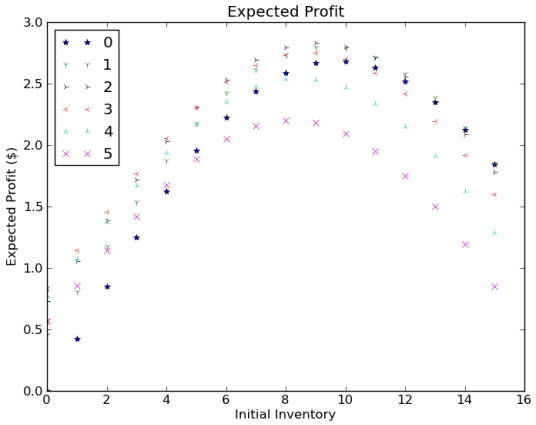

Figure 6.9: Expected profit in a monopolistic market with differentiated, substitutes. Let $\lambda_1(p_1, p_2) = 15 - 4p_1 + 1p_2$, $\lambda_2(p_1, p_2) = 10 + 2p_1 - 7p_2$, $C_{a,0} = 3.5$, $C_{b,0} = 1$, $C_{a,T} = 2$, and $C_{b,T} = 0.5$, and $T = 3$.

In Figure 6.9, we see that the firms maximum profit is near 2.83 when the inventory levels

are at $(I_{a,0}, I_{b,0}) = (2, 9)$. The combined total profit between the two firms in the duopolistic case were roughly 3.2, over 0.65 from firm a and about 2.5 from firm b. Again we see that a monopolistic market doesn't necessarily guarantee higher joint profits.

Given the optimal initial inventory levels, we can find the optimal sales price strategies, the cross price elasticities, the expected remaining revenues, and the progression of the inventory probabilities. We omit the expected remaining revenue and the inventory probabilities graphs as we don't gain any new insights from them.

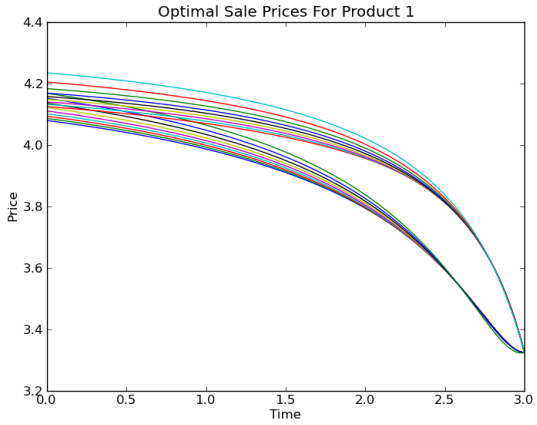

Figure 6.10: Optimal sales price strategy in a monopolistic market with differentiated, substitutes. Let $\lambda_1(p_1, p_2) = 15 - 4p_1 + 1p_2$, $\lambda_2(p_1, p_2) = 10 + 2p_1 - 7p_2$, $C_{a,0} = 3.5$, $C_{b,0} = 1$, $C_{a,T} = 2$, and $C_{b,T} = 0.5$, and $T = 3$.

Although the total profit is less in the monopolistic case, we see in Figures 6.10 and 6.11 that the optimal sales prices of the two goods are higher than in the duopolistic case. This is what we'd expect but we'd also expect higher profits as a whole. We will show in the complementary goods case that it is possible for the monopoly to charge less than the duopolistic firms.

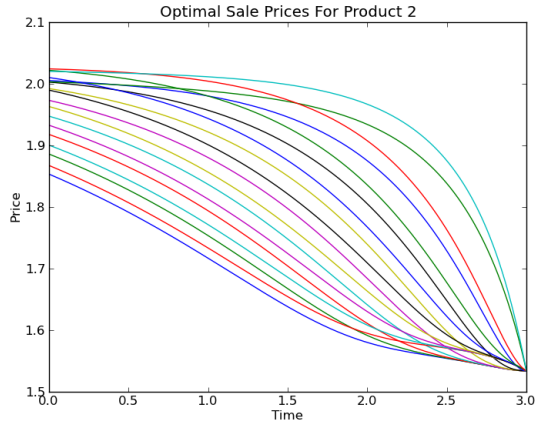

Figure 6.11: Optimal sales price strategy in a monopolistic market with differentiated, substitutes. Let $\lambda_1(p_1, p_2) = 15 - 4p_1 + 1p_2$, $\lambda_2(p_1, p_2) = 10 + 2p_1 - 7p_2$, $C_{a,0} = 3.5$, $C_{b,0} = 1$, $C_{a,T} = 2$, and $C_{b,T} = 0.5$, and $T = 3$.

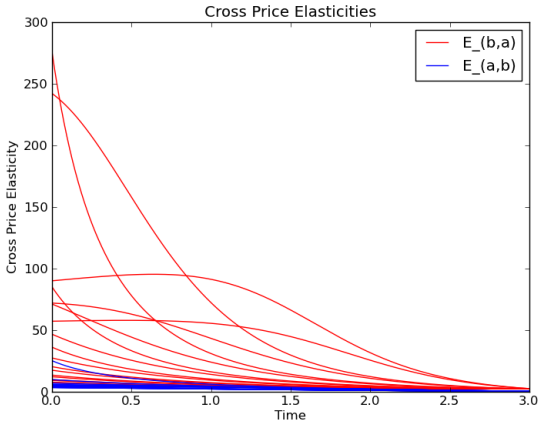

Figure 6.12: Cross price elasticity in a monopolistic market with differentiated, substitutes. Let $\lambda_1(p_1, p_2) = 15 - 4p_1 + 1p_2$, $\lambda_2(p_1, p_2) = 10 + 2p_1 - 7p_2$, $C_{a,0} = 3.5$, $C_{b,0} = 1$, $C_{a,T} = 2$, and $C_{b,T} = 0.5$, and $T = 3$.

6.2 Complements

We now examine what happens when the goods are complements. Again let the demand of the goods be random variables modeled with Poisson processes. The goods are complements and assume the demand for the Poisson rates is linear and given by the system of equations

below

$$\begin{bmatrix} \lambda_1(t) \\ \lambda_2(t) \end{bmatrix} = \begin{bmatrix} 15 \\ 20 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix}.$$

Let the initial and salvage costs of the goods be $C_{a,0} = 3$, $C_{b,0} = 1$, $C_{a,T} = 2$, and $C_{b,T} = 0.5$. Consider first a duopolistic model and then a monopolistic model.

6.2.1 Duopolistic. First consider a duopolistic market with two different firms the respective goods. Plot the expected profit to find the Nash equilibrium for initial inventory levels.

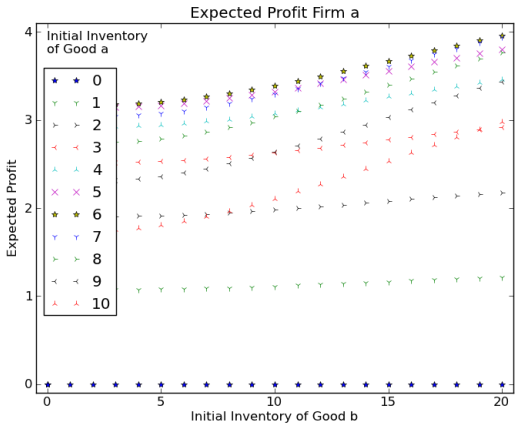

Figure 6.13: Expected profit of firm one in a duopolistic market with differentiated, complementary goods. Let $\lambda_1(p_1, p_2) = 15 - 3p_1 - 1p_2$, $\lambda_2(p_1, p_2) = 20 - 2p_1 - 7p_2$, $C_{a,0} = 3$, $C_{b,0} = 1$, $C_{a,T} = 2$, and $C_{b,T} = 0.5$, and $T = 3$.

In Figure 6.13, it shows that whenever firm b starts with less than 20 units of inventory, firm a's optimal initial inventory is six units. Figure 6.14 suggests that whenever firm a has six units of inventory, firm b maximizes optimal profit at initial inventory level of 13. So the Nash Equilibrium is found at $(I_{a,0}, I_{b,0}) = (6, 13)$. At the Nash Equilibrium, firm a has a profit of just under 3.56 and firm b has a profit of just under 6.33. So they have a combined total profit of about 9.89. Now use the previous techniques to find the optimal sales price strategies, the cross price elasticities, and the expected remaining revenues for both firms, a

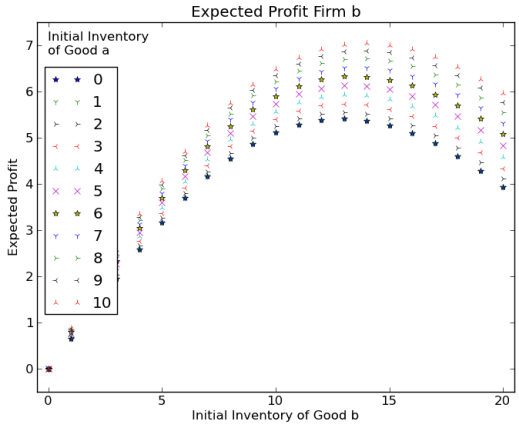

Figure 6.14: Expected profit of firm two in a duopolistic market with differentiated, complementary goods. Let $\lambda_1(p_1, p_2) = 15 - 3p_1 - 1p_2$, $\lambda_2(p_1, p_2) = 20 - 2p_1 - 7p_2$, $C_{a,0} = 3$, $C_{b,0} = 1$, $C_{a,T} = 2$, and $C_{b,T} = 0.5$, and $T = 3$.

and b.

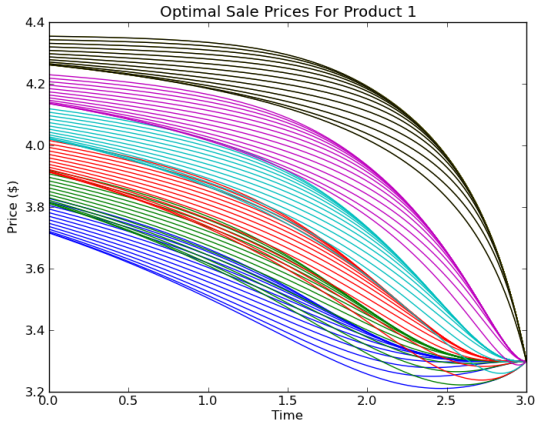

Figure 6.15: Optimal sales price strategy of firm one in a duopolistic market with differentiated, complementary goods. Let $\lambda_1(p_1, p_2) = 15 - 3p_1 - 1p_2$, $\lambda_2(p_1, p_2) = 20 - 2p_1 - 7p_2$, $C_{a,0} = 3$, $C_{b,0} = 1$, $C_{a,T} = 2$, and $C_{b,T} = 0.5$, and $T = 3$.

In Figures 6.15-6.19, the color lines correspond to the inventory levels of the corresponding firm as the levels of inventory of the other firm change.

Note again that the sales price of the second good, firm b, depend much more on the prices of firm a. Also see that sales price dips below the final sales price. This phenomenon happened in our previous complementary goods examples but is especially noticeable in Fig-

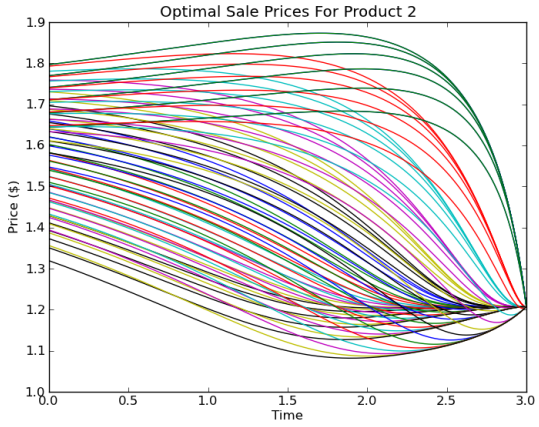

Figure 6.16: Optimal sales price strategy of firm two in a duopolistic market with differentiated, complementary goods. Let $\lambda_1(p_1, p_2) = 15 - 3p_1 - 1p_2$, $\lambda_2(p_1, p_2) = 20 - 2p_1 - 7p_2$, $C_{a,0} = 3$, $C_{b,0} = 1$, $C_{a,T} = 2$, and $C_{b,T} = 0.5$, and $T = 3$.

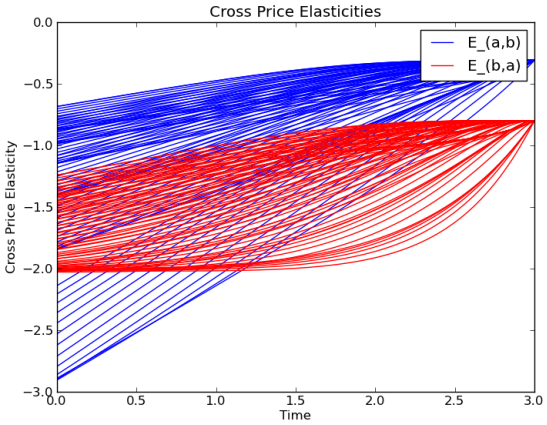

Figure 6.17: Cross price elasticities in a duopolistic market with differentiated, complementary goods. Let $\lambda_1(p_1, p_2) = 15 - 3p_1 - 1p_2$, $\lambda_2(p_1, p_2) = 20 - 2p_1 - 7p_2$, $C_{a,0} = 3$, $C_{b,0} = 1$, $C_{a,T} = 2$, and $C_{b,T} = 0.5$, and $T = 3$.

ure 6.19, the sales price for firm b.

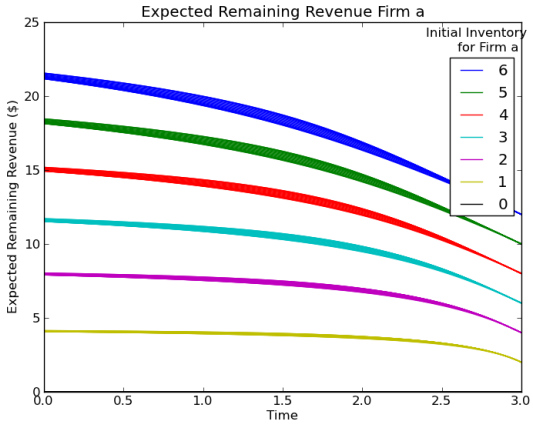

Figure 6.18: Expected remaining revenue of firm one in a duopolistic market with differentiated, complementary goods. Let $\lambda_1(p_1, p_2) = 15 - 3p_1 - 1p_2$, $\lambda_2(p_1, p_2) = 20 - 2p_1 - 7p_2$, $C_{a,0} = 3$, $C_{b,0} = 1$, $C_{a,T} = 2$, and $C_{b,T} = 0.5$, and $T = 3$.

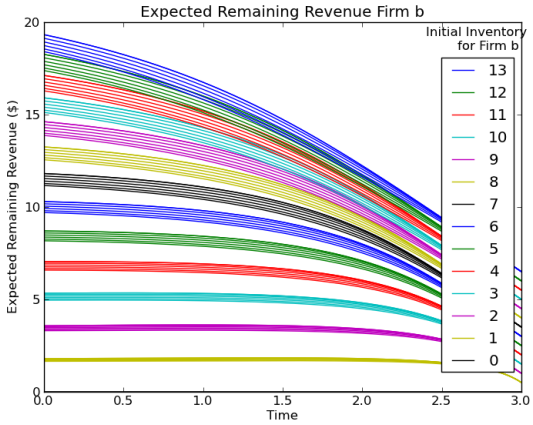

Figure 6.19: Expected remaining revenue of firm two in a duopolistic market with differentiated, complementary goods. Let $\lambda_1(p_1, p_2) = 15 - 3p_1 - 1p_2$, $\lambda_2(p_1, p_2) = 20 - 2p_1 - 7p_2$, $C_{a,0} = 3$, $C_{b,0} = 1$, $C_{a,T} = 2$, and $C_{b,T} = 0.5$, and $T = 3$.

6.2.2 Monopolistic. Now consider a monopolistic market with the same two goods. The expected profit is given in Figure 6.20 and shows that the optimal initial inventory level is $(I_{a,0}, I_{b,0}) = (7, 15)$. At the optimal initial inventory levels, the expected profit is just over 9.95. Generally we expect a monopoly to make a significantly more amount of money than a duopolistic market. It is interesting that this is only 0.06 above the combined profit for the duopolistic case; the expect profits of the two firms in the duopolistic case summed to roughly 9.89. This goes against our conventional wisdom but it makes sense because of the

cross price elasticity between the two goods. Because the goods are complements, we see that monopoly doesn't price gouge because otherwise, they wouldn't be able to much inventory. Again this reaffirms that the monopolistic's expected profit is not necessarily significantly larger than the combined profit in the duopolistic case. Now find the optimal sales price strategies and the cross price elasticities starting at the optimal initial inventory values.

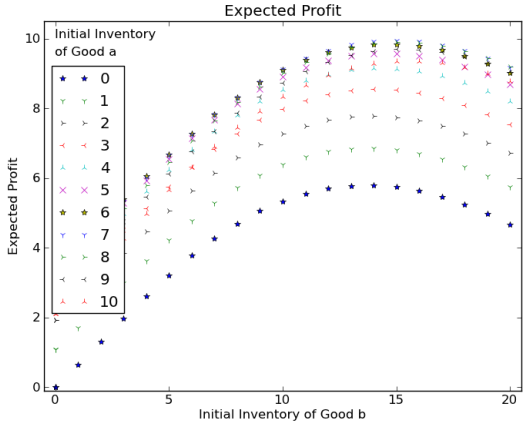

Figure 6.20: Expected profit of firm one in a monopolistic market with differentiated, complementary goods. Let $\lambda_1(p_1, p_2) = 15 - 3p_1 - 1p_2$, $\lambda_2(p_1, p_2) = 20 - 2p_1 - 7p_2$, $C_{a,0} = 3$, $C_{b,0} = 1$, $C_{a,T} = 2$, and $C_{b,T} = 0.5$, and $T = 3$.

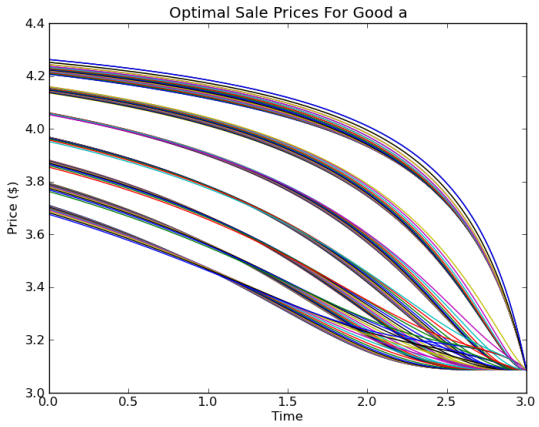

Figure 6.21: Optimal sales price strategy of firm one in a monopolistic market with differentiated, complementary goods. Let $\lambda_1(p_1, p_2) = 15 - 3p_1 - 1p_2$, $\lambda_2(p_1, p_2) = 20 - 2p_1 - 7p_2$, $C_{a,0} = 3$, $C_{b,0} = 1$, $C_{a,T} = 2$, and $C_{b,T} = 0.5$, and $T = 3$.

The interesting phenomenon, is that both sales prices of good a and b are both lower in the monopolistic case than in the duopolistic, as seen in Figures 6.15-6.16 and 6.21-6.22.

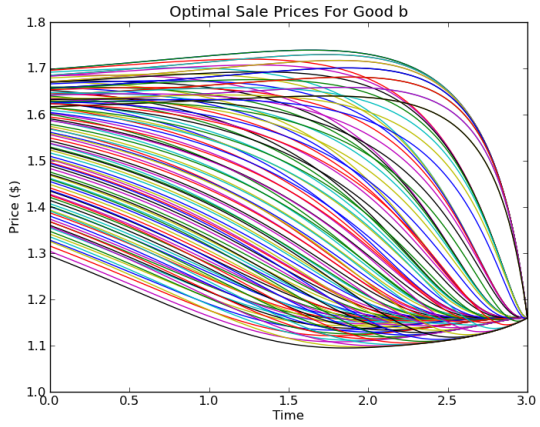

Figure 6.22: Optimal sales price strategy of firm two in a monopolistic market with differentiated, complementary goods. Let $\lambda_1(p_1, p_2) = 15 - 3p_1 - 1p_2$, $\lambda_2(p_1, p_2) = 20 - 2p_1 - 7p_2$, $C_{a,0} = 3$, $C_{b,0} = 1$, $C_{a,T} = 2$, and $C_{b,T} = 0.5$, and $T = 3$.

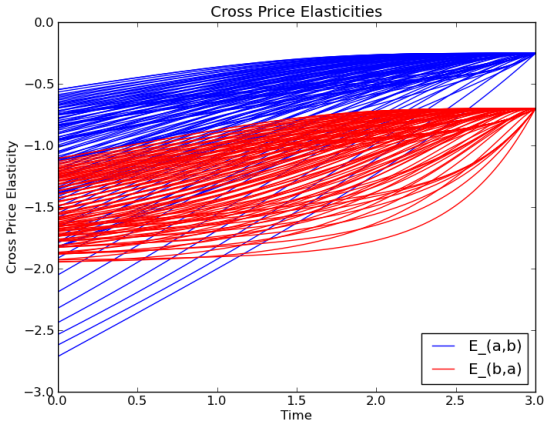

Figure 6.23: Cross price elasticities in a monopolistic market with differentiated, complementary goods. Let $\lambda_1(p_1, p_2) = 15 - 3p_1 - 1p_2$, $\lambda_2(p_1, p_2) = 20 - 2p_1 - 7p_2$, $C_{a,0} = 3$, $C_{b,0} = 1$, $C_{a,T} = 2$, and $C_{b,T} = 0.5$, and $T = 3$.

This is interesting because classical economic models tell us that monopolies always carry heavier prices, but this is not the case here. Because the goods are complements, the firm has incentive to keep the prices of one good low in order to sell the other good, regardless if the first good is low on inventory. Also Figures 6.17 and 6.23 show that the magnitudes of the cross price elasticity of demand for the monopolistic model are less than those of the duopolistic model. This is interesting and is most likely due to the fact that the monopoly has lower prices.

Chapter 7. Conclusions

Although the Bertrand model has been accepted for over one hundred years, there are some disadvantages to the model. Amongst others, the most poignant is that the Bertrand model is a deterministic model. We wanted to propose a model that was a probabilistic model for demand and one that allowed price to be above marginal cost. Using Poisson distributions as arrival rates for demand, we accomplished both goals by creating a probabilistic model that more accurately modeled the pricing behavior of retailers in oligopolistic markets on finite time horizons.

Throughout the several market structures we explored, we found several interesting phenomena worth noting, some harder to explain than others. First off, as a rule of thumb, we noticed that as time nears the expiration time, prices tend to go down in an attempt to dump inventory; however, when the goods were complements, we noticed that the optimal sales price can indeed drop below the optimal price at expiration. This causes the optimal price to increase on some subinterval during the whole time interval. This only happened when a good had significantly more units of inventory in comparison to the other good. This behavior was caused because the prices of the low inventory good went relatively high as they had low inventory, which caused the demand for the high inventory good to plummet causing the corresponding good to dramatically drop the prices in an attempt to sell some inventory. As the expiration time approached and the low inventory firm started to drop its price, the demand for the high inventory good started to rise again so it could increase its prices. Note that this only happened because the goods were complements and did not happen when the goods were substitutes.

Another phenomenon we observed was the progression of the cross price elasticities between goods. It was first seen in Section 4.2 but was probably best seen in Section 6. We saw that at the beginning of the time period, the magnitude of the cross price elasticity was

high and decreased as time continued. This is due to the fact that at the beginning of the sales period, the prices are usually at their highest which causes the demand for both goods to be at their lowest. If we recall the equation for cross price elasticity, $E_{a,b} = \frac{\partial \lambda_a}{\partial P_b} * \frac{P_b}{\lambda_a}$, it is clear that this causes the magnitude to be greatest at the beginning of the sales period.

In Section 4.3, we also saw that when goods are substitutes, they have higher expected profits when their competitor has less inventory; whereas when the goods are complements, the firms have higher expected profits when their competitor has more inventory. Since less inventory is synonymous with higher prices, this makes sense. As competitors have higher prices, substitutes perform better. However, when goods are complements, they perform better when their competitors have lower prices, or in other words, their competitors have high inventory.

The next three phenomena deal with the behavior of monopolies. First it was shown that the optimal initial inventory levels for monopolies aren't always less than that of the corresponding duopolies in the same market structure, contrary to what we'd expect. This was first shown in Section 5.2 where the optimal initial inventory levels for both substitutes and complements were the same and later shown in Section 6.1 where the monopoly's initial inventory levels were higher than those of the duopolies given the same market structure and complementary goods.

Generally we'd expect that a monopoly will have much higher expected profits than the combined total expected profits of the two duopolies in a market. We found that this wasn't always the case. In the homogeneous goods market (Section 5.2), we found that when the goods were substitutes, the total expected profits went roughly from 1.6 to 2.25 (over 40% increase) as the market changed from a duopolistic market to a monopolistic market; however, when the goods were complements, the total expected profits only went from around 0.75 to

0.8 (a 6.67% gain). Still a gain but not nearly as significant. In Sections 6.1 (differentiated substitutes), the total expected profits decreased from 3.2 to 2.83 (over 11% decrease) as the market switched from a duopoly to a monopoly. Whereas in Section 6.2 (differentiated complements), the total expected profits slightly increased from 9.89 to 9.95 (less than a 1% gain) with the switch.

We also expect the price of goods to increase in the presence of a monopoly. However that was not the case in Section 6.2 when the goods were differentiated complements. The sales prices of the monopoly were actually sightly lower than the prices of the duopolies. This happens because the monopoly has incentives to keep the price of one good low in order to keep demand for the other good high. So there's a relationship between prices and demand that keeps both prices low. This doesn't happen when the goods are substitutes because raising the price of one good will increase the demand for the other, which would be beneficial for the monopoly.

When the two goods are differentiated, there is also a disparity in the leverage the two firms have in the market as seen in Section 6. The differences in market leverage causes there to be imbalances in the market. There are several effects of the market imbalances seen in the behavior of the pricing strategies, the cross price elasticities, and the expected remaining revenue.

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